Practice problems: Even/odd proofs—Solutions

1. **Sums and products of even/odd numbers.** Prove the following statements, using only the definitions of even and odd integers, and paying particular attention to the write-up.

   (a) If n and m are both odd, then $n+m$ is even.

   (b) If n is odd and m is even, then $n+m$ is odd.

   (c) If n and m are both even, then $n+m$ is even.

   (d) If n and m are both odd, then nm is odd; otherwise, nm is even.

   Solution.

   (a) ***Proof of "n and m odd $\Rightarrow n+m$ even":***
   Suppose n and m are odd integers.
   Then $n = 2k+1$ and $m = 2l+1$ for some $k, l \in \mathbb{Z}$, by the definition of an odd integer.
   Therefore $n+m = (2k+1)+(2l+1) = 2(k+l+1)$.
   Since k and l are integers, so is $k+l+1$.
   Hence $n+m = 2p$ with $p = k+l+1 \in \mathbb{Z}$.
   By the definition of an even integer, this shows that $n+m$ is even. ■[1]

   (b)–(d): can be proved in an analogous way.

2. **Even/odd squares:** Prove the following: (Hint: A direct proof doesn't work very well (try it!), so try to use contraposition. You may use the statements established in Problem 1.)

   (a) Let n be an integer. If n^2 is odd, then n is odd.

   (b) Let n be an integer. If n^2 is even, then n is even.

   (c) Let n be an integer. Then n^2 is odd if and only if n is odd.

   Solution.

   (a) We prove the statement by contraposition. The negation of "odd" is "even" and the negation of "even" is "odd".[2] The contrapositive of statement (a) therefore is:

   (1) Let n be an integer. If n is even, then n^2 is even.

   Proof of (1):
   Assume n is an even integer.
   Then $n = 2k$ for some $k \in \mathbb{Z}$, by the definition of an even integer.
   Squaring, we get $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
   Since k is an integer, so is $2k^2$.
   Hence $n^2 = 2p$ with $p = 2k^2 \in \mathbb{Z}$.
   Therefore n^2 is even, by the definition of an even integer. ■

   (b) This can be proved in the same way as (a).

   (c) Here we have an "if and only if" statement, which breaks into the following two implications:

   - If $n \in \mathbb{Z}$ and n^2 is odd, then n is odd.
   - If $n \in \mathbb{Z}$ and n is odd, then n^2 is odd.

   The first of these implications, is the same as (a), while the second is the contraposition of (b), and hence equivalent to (b). Thus, (c) is a consequence of (a) and (b).

3. **Products and sums of perfect squares:**

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[1] The "end-of-proof box" is one of several common ways to indicate the end of a proof. (Another commonly used end-of-proof notation is "QED", an acronym for "quod erat demonstrandum".) The black box symbol is called the "Halmos symbol", named after the mathematician Paul Halmos (1916–2006). Halmos received his degree from the U of I in the 1930s and wrote about his life as a student here in his autobiography. See `http://www.math.uiuc.edu/~hildebr/halmosquotes0.html` for some choice quotes about Champaign and Urbana, Newman Hall, Illini Hall, Calculus, and some of his professors.

[2] That every integer is either even or odd, and that no integer is both even and odd, is not quite as obvious as it might seem, but rather a consequence of a general result about divisibility ("division algorithm") that will be established later, but which we will assume here.

(a) **Products:** Prove the following statement. (Hint: First rewrite this statement in more explicit form, using quantifiers and variables.)

(P) A product of two perfect squares is always a perfect square.

(b) **Sums:** Here the situation is more complicated. For example, the perfect squares 1^2 and 2^2 add up to 5, which is not a perfect square, so the sum analog of the above result is certainly not true. However, there are examples of perfect squares that add up to another prefect square: $3^2 + 4^2 = 5^2$ is the simplest, but there are a few others: $5^2 + 12^2 = 13^2$, $6^2 + 8^2 = 10^2$, $7^2 + 24^2 = 25^2$. However, you won't find any examples in which the two perfect squares on the left are both odd. Your task is to prove this, i.e.:

(S) A sum of two odd perfect squares is never a perfect square.

Solution:
Written more explicitly, the statements (P) and (S) become:

(1) *If s and t are perfect squares, then st is a perfect square.*

(2) *Suppose s and t are odd perfect squares. Then $s + t$ is not a perfect square.*

Proof of (1):
Suppose s and t are perfect squares.
Then $s = n^2$ and $t = m^2$ for some $n, m \in \mathbb{Z}$, by the definition of a perfect square.
Multiplying the two expressions for s and t, we get $st = (n^2)(m^2) = (nm)^2$.
Since n and m are integers, so is nm.
Hence $st = p^2$ with $p = nm \in \mathbb{Z}$.
Therefore st is a perfect square. ■

Proof of (2):
We prove (2) by the method of contradiction.
Suppose s and t are odd perfect squares, and assume that $(*)$ $s + t$ is also a perfect square.
Then $s = n^2$, $t = m^2$, and $s + t = p^2$, for some $n, m, p \in \mathbb{Z}$, by the definition of a perfect square.
Since $s = n^2$ and $t = m^2$ are odd squares, their squareroots, n and m, must be odd integers as well (by Problem 2).
Hence $n = 2k + 1$ and $m = 2l + 1$ for some $k, l \in \mathbb{Z}$, by the definition of an odd integer.
Moreover, since the sum of two odd numbers is even (by Problem 1), $s + t = p^2$ is even.
Hence, the squareroot of this number, p, must be even as well.
Therefore $p = 2h$ for some $h \in \mathbb{Z}$.
Thus we have

$$s = n^2 = (2k + 1)^2, \quad t = m^2 = (2l + 1)^2, \quad s + t = p^2 = (2h)^2.$$

Equating the two expressions for $s + t$ we get

$$\begin{aligned} (2k + 1)^2 + (2l + 1)^2 &= (2h)^2, \\ 4k^2 + 4k + 1 + 4l^2 + 4l + 1 &= 4h^2, \\ 4(k^2 + k + l^2 + l) + 2 &= 4h^2, \\ k^2 + k + l^2 + l - h^2 &= -\frac{1}{2}. \end{aligned}$$

In the latter equation the left side is an integer, while the right-hand side is not an integer.

Thus we have arrived at a contradiction.

Therefore our assumption that $s + t$ is a perfect square is false, and we have shown that if s and t are odd perfect squares, then $s + t$ cannot be not a perfect square. ■

4. **Solutions to quadratic equations:**

(a) Prove the following:
If a, b, c are odd *integers, then the equation $ax^2 + bx + c = 0$ has no integer solution.*

(b) Prove that the above result remains true if "integer solution" is replaced by "rational solution".

[This is the equation considered at the beginning of Chapter 2 of the text and it is a nice illustration of the power of "parity arguments". For the proof, you may use any of the properties of sums and products of even/odd integers established in Problem 1.]

Solution:
Proof of (a): *We use again the method of contradiction. Suppose x is an integer solution to $ax^2+bx+c=0$. We distinguish between two cases, x even and x odd; in each case, we will show that ax^2+bx+c must be odd and hence cannot equal 0, thus establishing a contradiction. (Note that 0 is even since it is of the form $2 \cdot k$ with $k=0$.) We will make repeated use of the properties of sums and products of odd/even numbers established in Problem 1.*

If x is even, then ax^2 and bx are both even since the product of an even integer with any integer is even. Since the sum of two even numbers is even, ax^2+bx is even as well. However, since, by assumption, c is odd and the sum of an odd and an even number is odd, ax^2+bx+c must be odd.

If x is odd, then ax^2 and bx are both odd since, by assumption, a and b are odd and the product of odd integers is odd. Since the sum of two odd numbers is even, ax^2+bx is even. Since, by assumption, c is odd and the sum of an odd and an even number is odd, we again conclude that ax^2+bx+c must be odd.

Thus, in either case, we obtain that ax^2+bx+c is odd and hence cannot be equal to 0. Therefore, our assumption that there is an integer solution to the equation $ax^2+bx+c=0$ is false. ■

Proof of (b): *Again we argue by contradiction. Suppose x is a rational solution to $ax^2+bx+c=0$. Since x is rational number, it can be written as $x=p/q$ for some integers p and q. We may assume the fraction p/q is in reduced form, i.e., the numerator and denominator have no common factors. Substituting $x=p/q$ into the equation $ax^2+bx+c=0$ and clearing denominators, we get*

$$a(p/q)^2 + b(p/q) + c = 0,$$
$$ap^2 + bpq + cq^2 = 0.$$

We now argue in a much the same way as above, considering the possible parities of p and q and showing that in each case, $ap^2+bqp+cq^2$ must be odd, and hence cannot equal 0.

Since we assumed p/q to be a reduced fraction, p and q cannot both be even, so we are left with three cases: (i) p even and q odd; (ii) p odd and q even; (iii) p odd and q odd.

In case (i), ap^2 is even, bpq is even, and cq^2 is odd, so $ap^2+bqp+cq^2$ is odd.

In case (ii), ap^2 is odd, bpq is even, and cq^2 is even, so $ap^2+bqp+cq^2$ is odd.

In case (iii), all three terms ap^2, bpq, and cq^2, are odd, so $ap^2+bqp+cq^2$ is odd as well.

Thus, in either case, we obtain that $ap^2+bqp+cq^2$ is odd and hence cannot be equal to 0. Therefore, our assumption that there is a rational solution to the equation $ax^2+bx+c=0$ is false. ■

5. **Irrationality proofs:** Prove that $\sqrt{2}$ is irrational. (Hint: Try contradiction.)

Proof: *Suppose the conclusion is false, i.e., assume that $\sqrt{2}$ is not irrational. We shall show that this leads to a contradiction.*

Our assumption that $\sqrt{2}$ is not irrational means that $\sqrt{2}$ is rational, i.e., of the form $\sqrt{2}=p/q$ for some integers p and q (where $q \neq 0$). We may assume that the fraction p/q is in reduced form, i.e., that p and q have no common factors. In particular, we know that ($*$) p and q cannot both be even.

Squaring each side of $\sqrt{2}=p/q$ and clearing denominators, we get $2=(p/q)^2=p^2/q^2$ and hence

$$2q^2 = p^2. \tag{1}$$

Since q^2 is an integer, $2q^2$ is even, so by (1), $p^2=2q^2$ must be even as well. Since (by Problem 2) the squareroot of an even perfect square is even, we conclude that p must be even. Hence $p=2k$ for some integer k.

Substituting this into (1), we get $2q^2=(2k)^2=4k^2$, and thus

$$q^2 = 2k^2. \tag{2}$$

But (2) implies that q^2 is even, and as before we conclude that q itself must be even.

Thus we have obtained that both p and q are even, contradicting ($*$). This contradiction shows that our original assumption, that $\sqrt{2}$ is not irrational, was false. Hence we have proved that $\sqrt{2}$ is irrational. ■.