## About this worksheet

In this worksheet you will practice constructing and writing up proofs of statements involving the parity (even or odd) of integers and related properties. In this context, the structure of the proofs becomes particularly simple, and the statements tend to be very intuitive (and some may seem obvious), so the focus should be on the write-up: Begin with scratch work to come up with the basic structure of the proof. Then write up the argument in a logical sequence of steps, using the proper definitions (which are recalled below), correct mathematical notation and terminology (in particular, use quantifiers when needed). Write in complete complete sentences and use English words, rather than logical symbols (such as $\forall$, $\exists$, $\Rightarrow$), in the final write-up.

## Definitions and notations[1]

- **Even integers:** An integer $n$ is called *even* if it is of the form $n = 2k$ for some $k \in \mathbb{Z}$. Denote the set of all even integers by $E$.
- **Odd integers:** An integer $n$ is called *odd* if it is of the form $x = 2k + 1$ for some $k \in \mathbb{Z}$. *An integer is either even or odd, but not both.*[2]
- **Divisibility:** Let $d$ be a nonzero integer. An integer $n$ is said to be *divisible by d* if it is of the form $n = dk$ for some $k \in \mathbb{Z}$. (Thus, an integer is even if and only if it is divisible by 2.)
- **Perfect square:** An integer $n$ is called a perfect square if it is of the form $n = k^2$ for some $k \in \mathbb{Z}$.
- **Rational and irrational numbers:** A real number $x$ is called *rational* if it can be written as $x = p/q$ with $p, q \in \mathbb{Z}$ and $q \neq 0$, and *irration* otherwise. The set of rational numbers is denoted by $\mathbb{Q}$; thus the set of irrational numbers is $\mathbb{R} - \mathbb{Q}$.

## Proof techniques:

- **Direct proof:** To prove $P \Rightarrow Q$ directly, assume $P$, then show that $Q$ is true.
- **Proof by contraposition:** To prove $P \Rightarrow Q$ by contraposition, assume $Q$ is false, then show that $P$ must be false. (This corresponds to proving the implication $\neg Q \Rightarrow \neg P$, which is logically equivalent to $P \Rightarrow Q$.)
- **Converse versus contraposition:** The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$. This is **not** the same as the contraposition $\neg Q \Rightarrow \neg P$. The latter is equivalent to $P \Rightarrow Q$, and hence a proof of the contrapositive statement is equivalent to a proof the original statement. By contrast, a proof of the converse $Q \Rightarrow P$ says nothing about the original statement; the converse may be true, while the original statement is not, and it may be false, while the original statement is true.
- **Proof by contradiction:** To prove a statement by contradiction, assume it is false, and derive from this assumption a contradiction, i.e., a statement such as $0 = 1$ that is patently false.

[1] **Meaning of "if" in definitions:** In a phrase like "$n$ is called *even* if ...", the word "if" has the meaning of a logical equivalence: "$n$ even $\Leftrightarrow$ ...", **not** a (one-directional) implication ("$n$ even $\Leftarrow$ ..."). (The latter interpretation wouldn't make sense in a definition context since it wouldn't be enough to define the property that $n$ is "even".)

[2] This is a non-trivial result that can be proved using number-theoretic techniques. For now, we simply assume this.

# Practice problems: Even/odd proofs

1. **Sums and products of even/odd numbers.** Prove the following statements, using only the definitions of even and odd integers, and paying particular attention to the write-up.
   (a) If $n$ and $m$ are both odd, then $n + m$ is even.
   (b) If $n$ is odd and $m$ is even, then $n + m$ is odd.
   (c) If $n$ and $m$ are both even, then $n + m$ is even.
   (d) If $n$ and $m$ are both odd, then $nm$ is odd; otherwise, $nm$ is even.

2. **Even/odd squares:** Prove the following: (Hint: A direct proof doesn't work very well (try it!), so try to use contraposition. You may use the statements established in Problem 1.)
   (a) Let $n$ be an integer. If $n^2$ is odd, then $n$ is odd.
   (b) Let $n$ be an integer. If $n^2$ is even, then $n$ is even.
   (c) Let $n$ be an integer. Then $n^2$ is odd if and only if $n$ is odd.

3. **Products and sums of perfect squares:**
   (a) **Products:** Prove the following statement. (Hint: First rewrite this statement in more explicit form, using quantifiers and variables.)

   *(P) A product of two perfect squares is always a perfect square.*

   (b) **Sums:** Here the situation is more complicated. For example, the perfect squares $1^2$ and $2^2$ add up to 5, which is not a perfect square, so the sum analog of the above result is certainly not true. However, there are examples of perfect squares that add up to another prefect square: $3^2 + 4^2 = 5^2$ is the simplest, but there are a few others: $5^2 + 12^2 = 13^2$, $6^2 + 8^2 = 10^2$, $7^2 + 24^2 = 25^2$. However, you won't find any examples in which the two perfect squares on the left are both odd. Your task is to prove this, i.e.:

   *(S) A sum of two odd perfect squares is never a perfect square.*

4. **Solutions to quadratic equations:**
   (a) Prove the following:
   *If $a, b, c$ are* odd *integers, then the equation $ax^2 + bx + c = 0$ has no integer solution.*
   (b) Prove that the above result remains true if "integer solution" is replaced by "rational solution".
   [This is the equation considered at the beginning of Chapter 2 of the text and it is a nice illustration of the power of "parity arguments". For the proof, you may use any of the properties of sums and products of even/odd integers established in Problem 1.]

5. **Irrationality proofs:** Prove that $\sqrt{2}$ is irrational. (Hint: Try contradiction.)