Induction Proofs, I: Basic Examples

A sample induction proof

Below is a complete proof of the formula for the sum of the first n integers, that can serve as a model for proofs of similar sum/product formulas.

We will prove by induction that, for all $n \in \mathbb{N}$,

$$(*) \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Base case: When $n = 1$, the left side of $(*)$ is 1, and the right side is $1(1+1)/2 = 1$, so both sides are equal and $(*)$ is true for $n = 1$.
Induction step: Let $k \in \mathbb{N}$ be given and suppose $(*)$ is true for $n = k$. Then

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \quad \text{(by induction hypothesis)} \\ &= \frac{k(k+1) + 2(k+1)}{2} \quad \text{(by algebra)} \\ &= \frac{(k+1)((k+1)+1)}{2} \quad \text{(by algebra).}\end{aligned}$$

Thus, $(*)$ holds for $n = k + 1$, and the proof of the induction step is complete.
Conclusion: By the principle of induction, $(*)$ is true for all $n \in \mathbb{N}$.

Notes

- **Begin any induction proof by stating precisely, and prominently, the statement ("$P(n)$") you plan to prove.** A good idea is to put the statement in a display and label it (e.g., by an asterisk $(*)$ as above), so that it is easy to spot, and easy to reference; see the sample proofs for examples.
- **Induction variable: n versus k.** Use the letter n for the variable in the statement of the formula (or proposition, etc.) that you seek to prove (and which, as pointed out above, you should have repeat at the beginning of every induction proof). Use k (or some other letter) for the variable appearing in the induction step. The reason for this distinction is that one can then say something like the following: "Let $k \in \mathbb{N}$ be given, and suppose $(*)$ is true for $n = k$. .... [Proof of induction step goes here] ... Therefore $(*)$ is true for $n = k + 1$."
- **The role of the induction hypothesis:** The induction hypothesis is the case $n = k$ of the statement we seek to prove ("$P(k)$"), and it is what you assume at the start of the induction step. You must get this hypothesis into play at some point during the proof of the induction step—if not, you are doing something wrong. The place where this hypothesis is used is the most crucial step in any induction argument, and you should clearly state, at the appropriate place, when you are using the induction hypothesis (e.g., "By the induction hypthesis we have ...", or as a parenthetical note "(by induction hypothesis)" in a chain of equations).

Practice problems

1. **Induction proofs, type I: Sum/product formulas:** The most common, and the easiest, application of induction is to prove formulas for sums or products of n terms. All of these proofs follow the same pattern.

   (a) $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$

   (b) $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ (sum of powers of 2)

   (c) $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ $(r \neq 1)$ (sum of finite geometric series)

   (d) $\sum_{i=0}^{n} i!i = (n+1)! - 1$.

2. **Induction proofs, type II: Inequalities:** A second general type of application of induction is to prove inequalities involving a natural number n. These proofs also tend to be on the routine side; in fact, the algebra required is usually very minimal, in contrast to some of the summation formulas.

   In some cases the inequalities don't "kick in" until n is large enough. By checking the first few values of n one can usually quickly determine the first n-value, say n_0, for which the inequality holds. Induction with $n = n_0$ as base case can then be used to show that the inequality holds for all $n > n_0$.

   (a) $2^n > n$

   (b) $2^n \geq n^2$ $(n \geq 4)$

   (c) $n! > 2^n$ $(n \geq 4)$

   (d) $(1-x)^n \geq 1 - nx$ $(0 < x < 1)$

   (e) $(1+x)^n \geq 1 + nx$ $(x > 0)$

3. **Induction proofs, type III: Extension of theorems from 2 variables to n variables:** Another very common and usually routine application of induction is to extend general results that have been proved for the case of 2 variables to the case of n variables. Below are some examples. In proving these results, use the case $n = 2$ as base case. To see how to carry out the general induction step (from the case $n = k$ to $n = k + 1$), it may be helpful to first try to see how get from the base case $n = 2$ to the next case $n = 3$.

   (a) Show that if $x_1, \dots, x_n$ are odd, then $x_1 x_2 \dots x_n$ is odd. (Use the fact (proved earlier) that the product of 2 odd numbers is odd, as starting point, and use induction to extend this result to the product of n odd numbers.)

   (b) Show that if a_i and b_i $(i = 1, 2, \dots, n)$ are real numbers such that $a_i \leq b_i$ for all i, then

   $$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i.$$

   (Use the fact (from Chapter 1) that $a \leq b$ and $c \leq d$ implies $a + c \leq b + d$.)

   (c) Show that if $x_1, \dots, x_n$ are real numbers, then

   $$\left| \sin\left(\sum_{i=1}^{n} x_i \right) \right| \leq \sum_{i=1}^{n} |\sin x_i| .$$

   (Use the trig identity for $\sin(\alpha + \beta)$.)

   (d) Show that if $A_1, \dots, A_n$ are sets, then

   $$(A_1 \cup \dots \cup A_n)^c = A_1^c \cap \dots \cap A_n^c.$$

   (This is a generalization of De Morgan's Law to unions of n sets. Use De Morgan's Law for two sets ($(A \cup B)^c = A^c \cap B^c$) and induction to prove this result.)