Practice problems Solutions

1. **Induction proofs, type I: Sum/product formulas:** The most common, and the easiest, application of induction is to prove formulas for sums or products of n terms. All of these proofs follow the same pattern.

   (a) $\sum_{i=1}^n i(i+1) = \frac{n(n+1)(n+2)}{3}$

   (b) $\sum_{i=0}^n 2^i = 2^{n+1} - 1$ (sum of powers of 2)

   (c) $\sum_{i=0}^n r^i = \frac{1-r^{n+1}}{1-r}$ $(r \neq 1)$ (sum of finite geometric series)

   (d) $\sum_{i=0}^n i!i = (n+1)! - 1$.

   Solution: *All proofs follow the pattern illustrated by the sample proof (of the formula $\sum_{i=1}^n i = n(n+1)/2$). We will carry out the details for (a) and (d). The other formulas can be proved similarly. (Note that (b) is a special case of (c).)*

   Proof of (a): *We seek to show that, for all $n \in \mathbb{N}$,*

   $$(*) \qquad \sum_{i=1}^n i(i+1) = \frac{n(n+1)(n+2)}{3}.$$

   Base case: *When $n = 1$, the left side of $(*)$ is $1 \cdot (1+1) = 2$, and the right side is $1 \cdot (1+1)(1+2)/3 = 2$, so both sides are equal and $(*)$ is true for $n = 1$.*

   Induction step: *Let $k \in \mathbb{N}$ be given and suppose $(*)$ is true for $n = k$. Then*

   $$\begin{aligned}\sum_{i=1}^{k+1} i(i+1) &= \sum_{i=1}^{k} i(i+1) + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \textit{(by induction hypothesis)} \\ &= \frac{(k+1)(k+2)(k+3)}{3}.\end{aligned}$$

   Thus, $(*)$ holds for $n = k+1$, and the proof of the induction step is complete.

   Conclusion: *By the principle of induction, it follows that $(*)$ is true for all $n \in \mathbb{N}$.*

   Proof of (d): *We seek to show that, for all $n \in \mathbb{N}$,*

   $$(*) \qquad \sum_{i=0}^n i!i = (n+1)! - 1.$$

   Base case: *When $n = 1$, the left side of $(*)$ is $0 + 1 \cdot 1! = 1$, and the right side is $(1+1)! - 1 = 1$, so both sides are equal and $(*)$ is true for $n = 1$.*

   Induction step: *Let $k \in \mathbb{N}$ be given and suppose $(*)$ is true for $n = k$. Then*

   $$\begin{aligned}\sum_{i=1}^{k+1} i \cdot i! &= \sum_{i=1}^{k} i \cdot i! + (k+1)(k+1)! \\ &= (k+1)! - 1 + (k+1)(k+1)! \quad \textit{(by induction hypothesis)} \\ &= (k+1)!(k+2) - 1 \\ &= (k+2)! - 1.\end{aligned}$$

   Thus, (2) holds for $n = k+1$, and the proof of the induction step is complete.

   Conclusion: *By the principle of induction, $(*)$ is true for all $n \in \mathbb{N}$.*

2. **Induction proofs, type II: Inequalities:** A second general type of application of induction is to prove inequalities involving a natural number n. These proofs also tend to be on the routine side; in fact, the algebra required is usually very minimal, in contrast to some of the summation formulas.

   In some cases the inequalities don't "kick in" until n is large enough. By checking the first few values of n one can usually quickly determine the first n-value, say n_0, for which the inequality holds. Induction with $n = n_0$ as base case can then be used to show that the inequality holds for all $n > n_0$.

   (a) $2^n > n$

   (b) $2^n \geq n^2$ $(n \geq 4)$

   (c) $n! > 2^n$ $(n \geq 4)$

   (d) $(1-x)^n \geq 1 - nx$ $(0 < x < 1)$

(e) $(1+x)^n \ge 1+nx$ $(x>0)$

Solution: *We will give detailed proofs for (c), (d), (e). The other inequalities can be proved similarly.*

Proof of (c): *A direct check of the inequality for the first few values of n shows that the left-right pairs in the stated inequality are $(1,2),(2,4),(6,8),(24,16),(120,32)$. Thus, the inequality fails for $n=1,2,3$, but holds for $n=4,5$. This suggests that it indeed holds for all n from 4 onwards. We will prove this by induction, i.e., we will show that*

$$(*) \qquad n! > 2^n$$

holds for all $n \ge 4$.

Base case: *For $n=4$, the left and right sides of $(*)$ are 24 and 16, respectively, so $(*)$ is true in this case.*

Induction step: *Let $k \ge 4$ be given and suppose $(*)$ is true for $n=k$. Then*

$$\begin{aligned}(k+1)! &= k!(k+1)\\ &> 2^k(k+1) \quad \text{(by induction hypothesis)}\\ &\ge 2^k \cdot 2 \quad \text{(since } k\ge 4 \text{ and so } k+1 \ge 2))\\ &= 2^{k+1}.\end{aligned}$$

Thus, $(*)$ holds for $n=k+1$, and the proof of the induction step is complete.

Conclusion: *By the principle of induction, it follows that $(*)$ is true for all $n \ge 4$.*

Proof of (d) and (e): *We will prove that for any real number $x > -1$*

$$(*) \qquad (1+x)^n \ge 1+nx.$$

holds for any $n \in \mathbb{N}$. This simultaneously proves both statements (d) and (e): (e) corresponds to the case $x>0$, while (d) corresponds to the case $-1<x<0$ (with $x'=-x$ in place of x).

Base case: *For $n=1$, the left and right sides of $(*)$ are both $1+x$, so $(*)$ holds.*

Induction step: *Let $k \in \mathbb{N}$ be given and suppose $(*)$ is true for $n=k$ and any real number $x>-1$. We seek to show that $(*)$ holds for $n=k+1$ and any real number $x>-1$.*

Let $x>-1$ be given. Then

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k(1+x)\\ &\ge (1+kx)(1+x) \quad \text{(by ind. hyp. and since } x>-1 \text{ and thus } (1+x)>0)\\ &= 1+(k+1)x+kx^2 \quad \text{(by algebra)}\\ &\ge 1+(k+1)x \quad \text{(since } kx^2 \ge 0).\end{aligned}$$

Hence $(*)$ holds for $n=k+1$, and the proof of the induction step is complete.

Conclusion: *By the principle of induction, it follows that $(*)$ holds for all $n \in \mathbb{N}$.*

3. **Induction proofs, type III: Extension of theorems from 2 variables to n variables:** Another very common and usually routine application of induction is to extend general results that have been proved for the case of 2 variables to the case of n variables. Below are some examples. In proving these results, use the case $n=2$ as base case. To see how to carry out the general induction step (from the case $n=k$ to $n=k+1$), it may be helpful to first try to see how get from the base case $n=2$ to the next case $n=3$.

(a) Show that if $x_1,\dots,x_n$ are odd, then $x_1x_2\dots x_n$ is odd. (Use the fact (proved earlier) that the product of 2 odd numbers is odd, as starting point, and use induction to extend this result to the product of n odd numbers.)

Solution: *We will prove by induction on n the following statement:*

$P(n)$: *If $x_1,\dots,x_n$ are odd numbers, then $x_1x_2\dots x_n$ is odd.*

We will use the following fact (proved earlier):

$(*)$ *If x and y are odd, then xy is odd.*

Base case: *For $n=1$, the product $x_1\dots x_n$ reduces to x_1, so is odd whenever x_1 is odd. Hence $P(1)$ is true.*

Induction step[1]:

> *Let $k \ge 1$, and suppose $P(k)$ is true, i.e., suppose that any product of k odd numbers is again odd.*
> *We seek to show that $P(k+1)$ is true, i.e., that any product of $k+1$ odd numbers is odd.*
> *Let $x_1,\dots,x_{k+1}$ be odd numbers.*
> *Applying the induction hypothesis to $x_1,\dots,x_k$, we obtain that the product $x_1x_2\dots x_k$ is odd.*
> *Since x_{k+1} is odd and, by $(*)$, the product of two odd numbers is again odd, it follows that $x_1x_2\dots x_{k+1} = (x_1\dots x_k)x_{k+1}$ is odd.*

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[1] To clearly display the structure of the argument, we have put each individual step on a separate line in this example. I recommend that you use a similar style in your homework and exam solutions.

This is what we wanted to show.

As $x_1, \dots, x_{k+1}$ were arbitrary odd numbers, we have proved $P(k+1)$, so the induction step is complete.

Conclusion: *By the principle of induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.*

Remark: *To show*

(b) Show that if a_i and b_i $(i = 1, 2, \dots, n)$ are real numbers such that $a_i \le b_i$ for all i, then

$$\sum_{i=1}^{n} a_i \le \sum_{i=1}^{n} b_i.$$

(Use the fact (from Chapter 1) that $a \le b$ and $c \le d$ implies $a + c \le b + d$.)

Solution: *We will prove by induction on n the following statement:*

$P(n)$: *For all real numbers a_i and b_i $(i = 1, \dots, n)$ such that $a_i \le b_i$ for all i we have*

$$(*) \qquad \sum_{i=1}^{n} a_i \le \sum_{i=1}^{n} b_i.$$

(Note that the quantifier "for all real numbers a_i and b_i" must be part of the induction statement we seek to prove.)

Base case: *For $n = 1$, the left and right sides are a_1 and b_1, respectively, and the inequality $(*)$ therefore follows from our hypothesis that $a_i \le b_i$ for all $i = 1, \dots, n$. Hence $P(1)$ is true.*

Induction step: *Let $k \ge 1$, and suppose $P(k)$ is true, i.e., suppose that $(*)$ holds for $n = k$ and any choice of real numbers $a_1, \dots, a_k$ and $b_1, \dots, b_k$ satisfying $a_i \le b_i$ for each i. We seek to show that $P(k+1)$ is true, i.e., that for any choice of real numbers $a_1, \dots, a_{k+1}$ and $b_1, \dots, b_{k+1}$ satisfying $a_i \le b_i$ for each i, the inequality $(*)$ holds.*

Let $a_1, \dots, a_{k+1}$ and $b_1, \dots, b_{k+1}$ be given real numbers such that $a_i \le b_i$ for each i. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} a_i &= \sum_{i=1}^{k} a_i + a_{k+1} \\
&\le \sum_{i=1}^{k} b_i + a_{k+1} \quad \textit{(by induction hypothesis applied to } a_1, \dots a_k\textit{)} \\
&\le \sum_{i=1}^{k} b_i + b_{k+1} \quad \textit{(by assumption } a_{k+1} \le b_{k+1}\textit{)} \\
&= \sum_{i=1}^{k+1} b_i.
\end{aligned}$$

Thus, $(*)$ holds for $n = k+1$, and since the $a_1, \dots, a_{k+1}$ and $b_1, \dots, b_{k+1}$ were arbitrary real numbers satisfying $a_i \le b_i$ for each i, we have obtained statement $P(k+1)$. Hence, the proof of the induction step is complete.

Conclusion: *By the principle of induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.*

(c) Show that if $x_1, \dots, x_n$ are real numbers, then

$$\left| \sin\left(\sum_{i=1}^{n} x_i \right) \right| \le \sum_{i=1}^{n} |\sin x_i| .$$

(Use the trig identity for $\sin(\alpha + \beta)$.)

Solution: *We seek to prove by induction on n the following statement:*

$P(n)$: *For all real numbers $x_1, \dots, x_n$ we have*

$$(*) \qquad \left| \sin\left(\sum_{i=1}^{n} x_i \right) \right| \le \sum_{i=1}^{n} |\sin x_i| .$$

The key to the argument is the trig identity

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \sin\beta \cos\alpha,$$

which is valid for any real α and β. Since $|\cos x| \le 1$, this identity implies, via the triangle inequality,

$$(**) \qquad \begin{aligned}
|\sin(\alpha + \beta)| &\le |\sin\alpha \cos\beta| + |\sin\beta \cos\alpha| \\
&\le |\sin\alpha| + |\sin\beta|.
\end{aligned}$$

Base case: *For $n = 1$, the left and right sides of $(*)$ are both equal to $|\sin x_1|$, so $(*)$ holds trivially in this case. Hence $P(1)$ is true.*

Induction step: *Let $k \ge 1$, and suppose $P(k)$ is true, i.e., suppose that $(*)$ holds for $n = k$ and any choice of real numbers $x_1, \dots, x_k$. We seek to show that $P(k+1)$ is true, i.e., that for any choice of real numbers $x_1, \dots, x_{k+1}$ the inequality $(*)$ holds.*

Let $x_1,\dots,x_{k+1}$ be given real numbers. Then

$$\begin{aligned}\left|\sin\left(\sum_{i=1}^{k+1}x_i\right)\right| &= \left|\sin\left(\left(\sum_{i=1}^{k}x_i\right)+x_{k+1}\right)\right|\\ &\le \left|\sin\left(\sum_{i=1}^{k}x_i\right)\right| + |\sin x_{k+1}| \quad \text{(by (**) with } \alpha=\sum_{i=1}^{k}x_i \text{ and } \beta = x_{k+1})\\ &\le \sum_{i=1}^{k}|\sin x_i| + |\sin x_{k+1}| \quad \text{(by induction hypothesis applied to } x_1,\dots,x_k)\\ &= \sum_{i=1}^{k+1}|\sin x_i|.\end{aligned}$$

Thus, (∗) holds for $n = k+1$, and since the $x_1,\dots,x_{k+1}$ were arbitrary real numbers, we have obtained statement $P(k+1)$. Hence, the proof of the induction step is complete.
Conclusion: *By the principle of induction, it follows that $P(n)$ is true for all $n\in\mathbb{N}$.*

(d) Show that if $A_1,\dots,A_n$ are sets, then

$$(A_1\cup\dots\cup A_n)^c = A_1^c\cap\dots\cap A_n^c.$$

(This is a generalization of De Morgan's Law to unions of n sets. Use De Morgan's Law for two sets ($(A\cup B)^c = A^c\cap B^c$) and induction to prove this result.)

Solution: *We seek to prove by induction on n the following statement:*

$P(n)$: *For all sets $A_1,\dots,A_n$ we have*

$$(A_1\cup\dots\cup A_n)^c = A_1^c\cap\dots\cap A_n^c. \tag{$*$}$$

The key to the argument is two set version of De Morgan's Law:

$$(A\cup B)^c = A^c\cap B^c, \tag{$**$}$$

which holds for any sets A and B.
Base case: *For $n=1$, the left and right sides of (∗) are both equal to A_1^c, so (∗) holds trivially in this case. Hence $P(1)$ is true.*
Though not absolutely necessary, we can also easily verify the next case, $n=2$: In this case, the left and right sides of (∗) are $(A_1\cup A_2)^c$ and $A_1^c\cap A_2^c$, respectively, so the identity is just the two set version of De Morgan's Law, i.e., (∗∗) with $A=A_1$ and $B=A_2$.
Induction step: *Let $k\ge 1$, and suppose $P(k)$ is true, i.e., suppose that (∗) holds for $n=k$ and any sets $A_1,\dots,A_k$. We seek to show that $P(k+1)$ is true, i.e., that for any sets $A_1,\dots,A_{k+1}$, (∗) holds.*
Let $A_1,\dots,A_{k+1}$ be given sets. Then

$$\begin{aligned}(A_1\cup\dots\cup A_{k+1})^c &= ((A_1\cup\dots\cup A_k)\cup A_{k+1})^c\\ &= (A_1\cup\dots\cup A_k)^c\cap A_{k+1}^c \quad \text{(by (**) with } A=(A_1\cup\dots\cup A_k) \text{ and } B=A_{k+1})\\ &= (A_1^c\cap\dots\cap A_k^c)\cap A_{k+1}^c \quad \text{(by induction hypothesis applied to } A_1,\dots,A_k)\\ &= A_1^c\cap\dots\cap A_k^c\cap A_{k+1}^c.\end{aligned}$$

Thus, (∗) holds for $n=k+1$, and since the $A_1,\dots,A_{k+1}$ were arbitrary sets, we have obtained statement $P(k+1)$. Hence, the proof of the induction step is complete.
Conclusion: *By the principle of induction, it follows that $P(n)$ is true for all $n\in\mathbb{N}$.*