Logical statements: Practice Problems

1. **Implications:** Express each of the following statements as a logical implication (e.g., $A \Leftarrow (\neg B)$) or equivalence (e.g., $A \Leftrightarrow B$). Also state its negation in English (in a form like "A is true, but B is false").

   (a) If A holds, then B holds.

   (b) A is true only if B is true.

   (c) A is true whenever B is true.

   (d) A is false only if B is false.

   (e) A is a necessary condition for B.

   (f) A is necessary and sufficient for B.

   (g) A holds if and only if B holds.

2. **Negations of English sentences.** Negate the following statements. Express the negations in English, *avoiding the use of words of negation when possible.*

   (a) All classroom have at least one chair that is broken.

   (b) No classroom has only chairs that are not broken.

   (c) Every student in this class has taken Math 231 or Math 241.

   (d) Every student in this class has taken Math 231 and Math 241.

   (e) In every section of Math 347 there is a student who has taken neither Math 231 nor Math 241.

3. **Negations of mathematical statements, I.** Translate the following sentences into logical notation, negate the statement using logical rules, then translate the negated statement back into English, *avoiding the use of words of negation when possible.* (Below f is a function from $\mathbb{R}$ to $\mathbb{R}$, and x_0 a given real number.)

   A bit harder, but very instructive: Many of the statements define familiar properties of functions (e.g., boundedness, monotonicity, etc.), or negations of such properties. Try to uncover these definitions and express in simple language the functions that are described by the statements.

   (a) $f(x, y) \neq 0$ whenever $x \neq 0$ and $y \neq 0$.

   (b) For all $M \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $|f(x)| \geq M$.

   (c) For all $M \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that for all $y > x$ we have $f(y) > M$.

   (d) For all $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that $f(y) > f(x)$.

   (e) For every $\epsilon > 0$ there exists $x_0 \in \mathbb{R}$ such that $|f(x)| < \epsilon$ for all $x > x_0$.

   (f) For every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

4. **Negations of mathematical statements, II.** This problem requires the formal definitions of a bounded set or function, and increasing, decreasing, nonincreasing, nondecreasing functions. These definitions can be found in Chapter 1 of the text and are collected below. (Here S is any set of real numbers, and f denotes a function from $\mathbb{R}$ to $\mathbb{R}$.)

   - S is *bounded* if there exists M such that $|x| \le M$ for all $x \in S$.
   - f is *bounded* if there exists M such that $|f(x)| \le M$ for all $x \in \mathbb{R}$.
   - f is *increasing* (or *strictly increasing*) if $f(x) < f(y)$ whenever $x < y$.
   - f is *nondecreasing* (or *weakly increasing*) if $f(x) \le f(y)$ whenever $x < y$.
   - f is *decreasing* (or *strictly decreasing*) if $f(x) > f(y)$ whenever $x < y$.
   - f is *nonincreasing* (or *weakly decreasing*) if $f(x) \ge f(y)$ whenever $x < y$.

   (a) Express the statement "f is *not bounded*" without using words of negation.

   (b) Express the statement "f is *not increasing*" (i.e., the negation of the "increasing" property) without using words of negation.

   (c) Compare the definitions of "nonincreasing" and "not increasing" (the latter being the negation of "increasing"). Does one imply the other? Are there functions that satisfy one property, but not the other?

5. **Practice with epsilon-delta definitions.** Recall the epsilon-delta definition of "$\lim_{x\to 0} f(x) = 0$":

(*) "For every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x)| < \epsilon$ whenever $|x| < \delta$."

The following statements are small perturbations of this definitions, some of which are equivalent to the original definition, while others are "botched" versions of this definition that have a drastically different meaning.

Which versions are equivalent to the continuity definition, and which are not?

Harder, but very instructive: For those definitions that are not equivalent to $\lim_{x\to 0} f(x) = 0$, try to determine, in as simple a language as possible, what they really define. Find examples (if they exist) of functions that satisfy the definition, and of functions that don't satisfy it. (Cf. Exercises 2.25–2.27 in the text for similar problems.)

(a) For every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, $|x| < \delta$ implies $|f(x)| < \epsilon$.

(b) For every $\delta > 0$ there exists $\epsilon > 0$ such that for all $x \in \mathbb{R}$, $|x| < \delta$ implies $|f(x)| < \epsilon$.

(c) There exists $\delta > 0$ such that for every $\epsilon > 0$ and for all $x \in \mathbb{R}$, $|x| < \delta$ implies $|f(x)| < \epsilon$.

(d) For every $\epsilon > 0$ and for all $x \in \mathbb{R}$ there exists $\delta > 0$ such that $|x| < \delta$ implies $|f(x)| < \epsilon$.