Practice Problems **Solutions**

1. Write the given sums and products explicitly without using sum/product notations, (e.g., as $1+2+\cdots+n$ or $\underbrace{1+\cdots+1}_{n}$), then evaluate them. (Here n is a positive integer.)

   (a) $\sum_{i=1}^n 1$ **Solution:** $\underbrace{1+1+\cdots+1}_{n}=n$

   (b) $\prod_{i=1}^n i$ **Solution:** $1\cdot 2\cdots n=n!$

   (c) $\prod_{i=1}^n 2$ **Solution:** $\underbrace{2\cdot 2\cdots 2}_{n}=2^n$

   (d) $\prod_{i=1}^n n^i$ **Solution:** $n^1\cdot n^2\cdots n^n=n^{1+2+\cdots+n}=n^{n(n+1)/2}$

   (e) $\sum_{i=1}^n n^i$ **Solution:** $n^1+n^2\cdots+n^n$. *By the geometric series formula this equals* $(n^{n+1}-n^1)/(n-1)$ *provided* $n\ge 2$.

   (f) $\prod_{i=1}^n \frac{n+i}{i}$ **Solution:** $\frac{(n+1)(n+2)\cdots(2n)}{1\cdot 2\cdots n}=\binom{2n}{n}$

2. Simplify, and evaluate if possible, the following sums by shifting the index or similar manipulations.

   (a) $\prod_{i=1}^n (i+2)$ **Solution:** *Set* $j=i+2$: $\prod_{j=3}^{n+2} j=(n+2)!/2!$

   (b) $\sum_{i=0}^n a_{i+1}$ **Solution:** *Set* $j=i+1$: $\sum_{j=1}^{n+1} a_j$

   (c) $\prod_{i=1}^n (n-i+1)$ **Solution:** *Set* $j=n-i+1$: $\prod_{j=1}^n j=n!$.

   (d) $\sum_{i=0}^n (a_{i+1}-a_i)$

   Solution:

   $$\sum_{i=0}^n (a_{i+1}-a_i)=\sum_{i=0}^n a_{i+1}-\sum_{i=0}^n a_i=\sum_{j=1}^{n+1} a_j-\sum_{i=0}^n a_i=a_{n+1}-a_0.$$

   Remark: *Sums of the above type are called* ***telescoping*** *since all terms except for the first and last one cancel.*

   (e) $\sum_{i=1}^n (4i-1)$

   Solution:

   $$\sum_{i=1}^n (4i-1)=4\sum_{i=1}^n i-\sum_{i=1}^n 1=4\frac{n(n+1)}{2}-n.$$

   (f) $\sum_{k=0}^\infty \frac{k+1}{k!}$ (Hint: Break the sum into two parts.)

   Solution:

   $$\sum_{i=0}^\infty \frac{i+1}{i!}=\sum_{i=0}^\infty \frac{i}{i!}+\sum_{i=0}^\infty \frac{1}{i!}=\sum_{i=1}^\infty \frac{1}{(i-1)!}+\sum_{i=0}^\infty \frac{1}{i!}$$
   $$=\sum_{j=0}^\infty \frac{1}{j!}+\sum_{i=0}^\infty \frac{1}{i!}=e+e=2e.$$

3. Write the following in summation notation and evaluate using appropriate summation formulas.

   (a) $3+7+11+\cdots+(4n-1)$ **Solution:** $\sum_{i=1}^n (4i-1)=4\frac{n(n+1)}{2}-n$.

   (b) $y^n+xy^{n-1}+x^2y^{n-2}+\cdots+x^{n-1}y+x^n$ $(x,y\ne 0)$.

   Solution: *In summation notation, the sum is* $\sum_{i=0}^n x^iy^{n-i}$. *If* $x\ne y$, *the latter sum can be evaluated by writing it as* $y^n\sum_{i=0}^n (x/y)^i$ *and applying the formula for the sum a finite geometric series, to get* $y^n(1-(x/y)^{n+1})/(1-(x/y))$. *If* $x=y$, *the sum can be evaluated directly: In this case, the terms are constant and equal to* x^n, *and since there are* $n+1$ *such terms, the sum equals* $(n+1)x^n$.