Math 361 X1 **Spring 2003**

Solution to the Car/Goat Problem

One of three doors hides a car (all three equally likely) and the other two hide goats. You choose Door 1. The host, who knows where the car is, then opens one of the other two doors to reveal a goat, and asks whether you wish to switch your choice. Say he opens Door 3; should you stick with your original choice, Door 1, or switch to Door 2?

Solution:

Relevant events: For $i = 1, 2, 3$, let C_i denote the event "car hides behind door i", let H_i be the event "host opens door i", and let W be the event that you win.

To compute: $P(W|H_3)$, for each of the two strategies (a) and (b). (There are two problems here, each requiring a separate argument.)

Given data: Since the car is behind each of the three doors with equal probability (assumption (1)), we have $P(C_i) = 1/3$ for $i = 1, 2, 3$. By assumption (2) the host always opens either door 2 or door 3. Thus, $H_2 = H_3^c$. By assumption (3), the host opens door 3 if the car is behind door 2, and door 2 if it is behind door 3, and he opens doors 3 and 2 with probabilities 1/2 each if it is behind door 1. Thus we have (i) $P(H_3|C_2) = 1$, (ii) $P(H_2|C_3) = 1$, (iii) $P(H_3|C_1) = q$, (iv) $P(H_2|C_1) = 1 - q$. From (ii) and complement formula for conditional probabilities it follows that (v) $P(H_3|C_3) = P(H_2^c|C_3) = 0$.

Computation:

(a) (switching strategy): In that case $P(W|H_3) = P(C_2|H_3)$ since, given that the host has opened door 3, you switch to door 2 and therefore win if and only if the car is behind door 2. By Bayes' rule with C_1, C_2, C_3 as partition of Ω

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\begin{aligned}
P(W|H_3) = P(C_2|H_3) &= \frac{P(H_3|C_2)P(C_2)}{P(H_3)} \\
&= \frac{P(H_3|C_2)P(C_2)}{P(H_3|C_1)P(C_1) + P(H_3|C_2)P(C_2) + P(H_3|C_3)P(C_3)}.
\end{aligned}
$$

Substitution $P(C_i) = 1/3$ and the values of $P(H_3|C_i)$ from (i), (iii), and (iv) above, we get

$$P(W|H_3) = \frac{1 \cdot (1/3)}{(1/2) \cdot (1/3) + 1 \cdot (1/3) + 0 \cdot (1/3)} = \frac{2}{3}.$$

(b) (no-switching strategy): Under strategy (b), you win if and only if the car hides behind door 1. Thus, $P(W|H_3) = P(C_1|H_3)$, and by Bayes' rule

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\begin{aligned}
P(W|H_3) = P(C_1|H_3) &= \frac{P(H_3|C_1)P(C_1)}{P(H_3)} \\
&= \frac{P(H_3|C_1)P(C_1)}{P(H_3|C_1)P(C_1) + P(H_3|C_2)P(C_2) + P(H_3|C_3)P(C_3)} = \frac{1}{3}.
\end{aligned}
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Conclusion: Under the switching strategy, the probability of winning is 2/3; under the no-switching strategy, the probability of winning is 1/3. Thus, you should switch.