Math 361, Spring 2003
Double Integrals

Tips on handling double integrals

- **Determining the limits of integration.** To evaluate a double integral over a given region R, the integral has to be written as an iterated integral with concrete limits on the x- and y-integrals. To determine these limits, if they are not already given, proceed as follows:
  - Draw a picture of the region of integration.
  - Referring to this picture, try to express the region in the form of two inequalities on x and y, with constant limits on one variable, and (possibly) variable limits on the other variable; e.g.: $0 \leq x \leq 1$, $1 - x \leq y \leq 1$.
  - Write the integral as an iterated integral, making sure that the outer integral has constant limits. (This usually dictates the order of the variables.)

- **Order of integration.** In most cases, a double integral can be written as an iterated integral in two ways, with the x-integration outside and the y-integration inside, and vice versa. However, to change the order of integration, one (usually) cannot simply swap the integration signs and the differentials dx and dy since this might result in outer integrals involving variable limits. Instead, one has to re-express the region in terms of inequalities, making sure that the outer variable is the one with constant limits and the inner variable has (possibly) variable limits. In some cases, changing the order of integration yields an integral that is somewhat easier to compute, but for the integrals that typically arise in actuarial exam problems it's probably not worth the trouble to change the order of a given integral.

- **Some useful formulas:** The following are some frequently occurring integrals. They are easy to evaluate directly, but knowing these formulas saves valuable time in an exam setting.

$$\int_0^1 x^c dx = \frac{1}{c+1}, \qquad \int_0^L x^c dx = \frac{L^{c+1}}{c+1} \quad (c > -1)$$
$$\int_1^\infty \frac{1}{x^c} dx = \frac{1}{c-1}, \qquad \int_L^\infty \frac{1}{x^c} dx = \frac{L^{1-c}}{c-1} \quad (c > 1)$$
$$\int_0^\infty e^{-cx} dx = \frac{1}{c}, \qquad \int_L^\infty e^{-cx} dx = \frac{e^{-cL}}{c} \quad (c > 0)$$

Practice Problems

1. In each of the following problems a region R is given. Express the integral $\iint_R f(x,y)dxdy$ over this region as an iterated integral in the form $\int_*^* \int_*^* f(x,y)dxdy$ and/or $\int_*^* \int_*^* f(x,y)dydx$ with specific limits in place of the asterisks.

   (a) The region given by $0 < x < 1, x < y < x + 1$

   (b) The part of the unit square $0 \le x \le 1, 0 \le y \le 1$, on which $y \le x/2$.

(c) The part of the unit square on which $x + y > 0.5$.

(d) The part of the unit square on which **both** x and y are greater than 0.5.

(e) The part of the unit square on which **at least one of** x and y is greater than 0.5. (This requires a sum of two integrals of the specified form.)

(f) The part of the region given by $0 < x < 50 - y < 50$ on which **both** x and y are greater than 20.

2. Change the order of integration in the following integrals. (Be sure to first sketch the region of integration.)

   (a) $\int_0^1 \int_{1-x}^1 f(x,y)dydx$

   (b) $\int_0^\infty \int_{2x}^\infty f(x,y)dydx$

3. Evaluate the following integrals.

(a) $\iint_R e^{-x-y} dxdy$, where R is the region in the first quadrant in which $x + y \leq 1$.

(b) $\iint_R e^{-x-2y} dxdy$, where R is the region in the first quadrant in which $x \leq y$

(c) $\iint_R (x^2 + y^2) dxdy$, where R is the region $0 \leq x \leq y \leq L$

(d) $\int\int_R f(x,y)dxdy$, where R is the region inside the unit square in which both coordinates x and y are greater than 0.5.

(e) $\int_0^1 \int_0^1 x \max(x,y)dydx$