Math 361 Exam 2 Solutions and Comments Spring 2003

Exam statistics:

(See the course web page for more details and grading information.)

Highest scores: 47, 44, 42, 41, 40
Lowest scores: 8, 9
Average score: 30
Curve: Cutoffs for A/B/C/D were set at 40/30/20/10 points, respectively.

Problem 1.

A box contains 100 computer chips, of which exactly 5 are good and the remaining 95 are bad.
(a) (5) Suppose first you take out chips one at a time (without replacement) and test each chip you have taken out until you have found a good one. Let X be the number of chips you have to take out in order to find one that is good. (Thus, X can be as small as 1 and as large as 96.) Find the distribution of X.
(b) (5) (This part is independent of part (a).) Suppose now that you take out exactly 10 chips and then test each of these 10 chips. Let Y denote the number of good chips among the 10 you have taken out. Find the distribution of Y.

Solution.

(a) **Values of X:** $X = 1, 2, \dots, 96$.
Computation of probabilities: The event $X = n$ means by definition that the nth chip taken out is the first good one, or, equivalently, that the first $n-1$ chips are bad, but the nth chip is good. The total number of ways to draw n chips out of 100 **without replacement and taking order into account** is $100 \cdot 99 \cdots (100 - n + 1)$. The number of such draws in which the first $n-1$ chips are among the 95 bad chips and the nth is among the 5 good chips is $95 \cdot 94 \cdots (95 - n + 2) \cdot 5$. Thus,

$$P(X = n) = \frac{95 \cdot 94 \cdots (95 - n + 2) \cdot 5}{100 \cdot 99 \cdots 100 - n + 1} = \frac{(95)_{n-1} \cdot 5}{(100)_n} \quad (n = 1, 2, \dots, 96).$$

Comments: It is essential to count **ordered** samples to come up with the above answer. With **unordered** samples one would not be able to take into account the fact that the good chip must be the **last** of the n chips checked. An answer like $\binom{5}{1}\binom{95}{n-1}/\binom{100}{n}$ would only count the probability that among n chips exactly one (but not necessarily the last one) is good. (However, dividing the latter formula by n does give the correct answer, as the factor $1/n$ compensates for the n different positions that the good chip can be at.)
(b) **Values of Y:** $Y = 0, 1, 2, \dots, 5$.
Computation of probabilities: The event $Y = n$ means that of the 10 chips exactly n are good. This is a standard box/ball type problem with the 5 good chips corresponding to 5 red balls and the 95 bad chips corresponding to 95 black balls. The probability $P(Y = n)$ is the same as probability that in a sample of 10 balls without replacement exactly n balls are red, i.e., we have

$$P(Y = n) = \frac{\binom{5}{n}\binom{95}{10-n}}{\binom{100}{10}} \quad (n = 0, 1, \dots, 5).$$

Comment: Taking out 10 chips is equivalent to drawing 10 balls without replacement. (It is clear from the wording of the problem that the chips taken out are not put back into the box.) The appropriate model here is that of a box/ball (or committee) type problem. A success/failure model would be quite inappropriate here since the proportion of good chips (and hence the probability of getting a good chip) changes every time a chip is taken out.

Problem 2.

A die is rolled repeatedly. Let X denote the number of the roll at which the **third** six occurs.
(a) (5) Find $P(X > 361)$ *without using the result of part (b).* Your answer should be such that it could be easily computed with a basic calculator. A sum involving a large number of terms would not qualify.
(b) (5) Find the distribution of X.

Solution.

(a) The event $X > 361$ means, by the definition of X, that the third six occurs after roll 361. But this is equivalent to saying that there are at most 2 sixes in the first 361 rolls. The probability for this event is a standard success/failure probability, namely

$$P(\leq 2 \text{ sixes in 361 rolls}) = \left(\frac{5}{6}\right)^{361} + \binom{361}{1}\left(\frac{5}{6}\right)^{360}\left(\frac{1}{6}\right) + \binom{361}{2}\left(\frac{5}{6}\right)^{359}\left(\frac{1}{6}\right)^{2}.$$

Comment: The key to this problem is the rephrasing of the event "$X > 361$" as "at most two sixes in the first 361 rolls". This type of reasoning has come up before (e.g., in Problem 5 of HW 3, or in connection with the birthday problem). Of course, the probability $P(X > 361)$ can be written as a sum of the 361 probabilities $P(X = n)$ for $n = 1, 2, \dots, 361$, but this is not a practical way to solve the problem, and it is an acceptable solution.
(b) **Values of X:** $3, 4, \dots$.
Computation of probabilities: For $n = 3, 4, \dots,$ we have

$$\begin{aligned} P(X = n) &= P(\text{third six occurs at trial } n) \\ &= P(\text{exactly two sixes in trials } 1, 2, \dots, n-1 \text{ and one six in trial } n) \\ &= \binom{n-1}{2}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{n-3}\left(\frac{1}{6}\right). \end{aligned}$$

Comment: An answer like $P(X = n) = \binom{n}{3}(1/6)^3(5/6)^{n-3}$ would be completely off target, as this is a straight S/F probability, namely the probability for getting three sixes in n rolls, which does not take into account the requirement that the third six has to occur at the n-th roll. (This fallacy come up in several homework problems and was pointed out repeatedly in that context.)

Problem 3.

Let $S = X + Y$ and $D = X - Y$, where X and Y are two independent random variables, each uniformly distributed on the set $\{0, 1, 2, 3, 4, 5\}$.
(a) (4) Find $E(S)$ and $E(D)$ *without computing the distribution of S and D.*
(b) (3) Find $E(S^2)$ *without computing the distribution of S.*
(c) (3) Determine, with rigorous justification, whether S and D are independent.

Solution.

(a) We first compute the expectations of X and Y By the given uniform distribution, X and Y have values $0, 1, 2, 3, 4, 5$ with probabilities $1/6$ each, so

$$E(X) = \sum_{\text{all } x} xP(X = x) = 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3\frac{1}{6} + 4\frac{1}{6} + 5\frac{1}{6}. = 2.5,$$

and similarly $E(Y) = 2.5$. We now apply the addition formula for the expection to obtain $E(S) = E(X+Y) = E(X) + E(Y) = 5$ and $E(D) = E(X - Y) = E(X) - E(Y) = 0$.
(b) Using again the properties of the expectation, we have $E(S^2) = E((X+Y)^2) = E(X^2)+2E(XY)+E(Y^2)$. By the formula for $E(g(X))$,

$$E(X^2) = \sum_{\text{all } x} x^2P(X = x) = 0^2 \cdot \frac{1}{6} + 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2\frac{1}{6} + 4^2\frac{1}{6} + 5^2\frac{1}{6} = \frac{55}{6}.$$

and similarly $E(Y^2) = 55/6$. Also, since X and Y are independent, $E(XY) = E(X)E(Y) = 2.5^2 = 25/4$. Hence, $E(S^2) = (55/6) + 2(25/4) + (55/6) = (55/3) + (25/2) = 185/6 = 30.833\ldots$

Comment: Problems (a) and (b) were meant to be exercises in applying rules for expectation, by writing $S = X + Y$ and expanding $(X + Y)^2$ to reduce the computation of $E(S)$ and $E(S^2)$ to that of $E(X), E(Y)$, $E(X^2), E(Y^2)$, which is straightforward and easy to do by hand. A direct computation of $E(S)$ (or $E(S^2)$) via the distribution of S and the formula $E(S) = \sum_s P(S = s)$ would be very messy and not doable by hand within the time limits of the exam; this approach was explicitly disallowed in the problem.

(c) The random variables S and D are **not** independent. For example, $P(S = 0, D = 0) = P(X + Y = 0, X - Y = 0) = P(X = 0, Y = 0) = (1/6)^2 = 1/36$ (using the independence of X and Y), which is not equal to $P(S = 0)P(D = 0)$ since $P(S = 0) = P(X = 0, Y = 0) = 1/36$ but $P(D = 0) = P(X = Y) < 1$.

Comment: If two r.v.'s X and Y are independent, they satisfy $(*)$ $E(XY) = E(X)E(Y)$. However, the converse is not true: If $(*)$ holds, it does not **necessarily** follow that X and Y are independent. Here, S and D happen to satisfy $(*)$, yet they are not independent.

Problem 4.

An instructor wants to create an exam consisting of 5 problems and covering 6 sections of the text. To this end, he first makes up 4 problems for each of the 6 sections, and then selects at random 5 (different) problems from these 24 problems.

(a) (5) What is the probability that the problems on the exam are all from different sections (i.e., that no section has more than one problem on the exam)?

(b) (5) What is the expected number of sections from which there is a problem on the exam?

Solution.

(a) [This problem is of just like as the "8 days of the year" problem discussed in class or some problems of similar type in HW 5] The total number of ways of selecting 5 (different) problems out of 24, taking order into account, is $\#(\Omega) = 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20$. The number of such selections in which each problem is from a different section is $24 \cdot 20 \cdot 16 \cdot 12 \cdot 8$ since there are 24 choices for the first problem, 20 for the second problem (since it can't be one of the four problems from the section of problem 1), 16 for the third problem, etc. Thus, the probability that all problems are from a different sections is

$$\frac{24 \cdot 20 \cdots 8}{24 \cdot 23 \cdots 20}.$$

[An alternative approach is to count unordered samples. Then $\#(\Omega) = \binom{24}{5}$, and $\#(A) = \binom{6}{5} \cdot 4^5$ (pick 5 sections out of 6, then pick from each of these 5 sections 1 out of 5 problems). The resulting probability, $\binom{6}{5}4^5/\binom{24}{5}$, is the same as the one obtained above.]

(b) We need to compute $E(X)$, where X is the number of sections that have a problem on the exam. The only feasible way to do this is by the indicator method, since the individual probabilities $P(X = x)$ are very difficult to compute (and nobody who tried this during the exam succeeded in doing this correctly). We define the events

$$A_i = \text{"Section } i \text{ has a problem on the exam"} \qquad (i = 1, 2, \ldots, 6).$$

Then X is equal to the number of A_i's that occur, so by the indicator method we have

$$E(X) = \sum_{i=1}^{6} P(A_i).$$

To compute $P(A_i)$, we use (as in the above-mentioned problems) the complement trick:

$$\begin{aligned} P(A_i) &= 1 - P(A_i^c) = 1 - P(\text{no problem from section } i \text{ is on the exam}) \\ &= 1 - P(5 \text{ chosen problems are among the 20 (out of 24) not in section } i) \\ &= 1 - \frac{20 \cdot 19 \cdots 16}{24 \cdot 23 \cdots 20} \end{aligned}$$

so

$$E(X) = 6\left(1 - \frac{20\cdot 19\cdots 16}{24\cdot 23\cdots 20}\right) = 6\cdot\frac{1125}{1771} = 3.81141.$$

[Alternatively, one can compute $P(A_i)$ directly, using a box/ball model, adding up the cases where Section i has 1, 2, 3, and 4 exam problems: This gives

$$P(A_i) = \frac{1}{\binom{24}{5}}\sum_{k=1}^{4}\binom{4}{k}\binom{20}{5-k},$$

which looks much more complicated, but is in fact equal to the expression obtained above.]

Comment: This problem was meant to be an exercise in applying the indicator method for computing expectations. The indicator method is an important method that complements the two other methods we discussed in class (via the distribution of X and the formula $(*)$ $E(X) = \sum_x xP(X = x)$, and via the rules for expectations – both of these methods were required for Problem 3), and applies in situations where the other methods fail completely. This is the case here. Computing $E(X)$ via the definition $(*)$ would require computing the probabilities $P(X = x)$ for all values $x = 1, 2, \ldots, 6$; as pointed out in class in connection with the birthday problem (which is essentially the same as this problem with the 365 possible birthdays corresponding to the 6 possible numbers showing up), computing these probabilities is exceedingly complicated, and there are no simple formulas for these probabilities. (In fact, this was an Extracredit problem in HW 6.)

Problem 5.

Suppose that X has Poisson(μ) distribution and that Y has geometric(p) distribution and is independent of X. Find simple formulas in terms of μ and p for the following probabilities. (The formulas should not involve an infinite sum.)

(a) (5) $P(X + Y = 2)$

(b) (5) $P(Y > X)$

Solution.

(a) By the independence of X and Y and the given Poisson resp. geometric distributions, the joint distribution of X and Y is

$$P(x, y) = P(X = x)P(Y = y) = e^{-\mu}\frac{\mu^x}{x!}p(1-p)^{y-1} \qquad (x = 0, 1, 2, \ldots;\ y = 1, 2, \ldots).$$

Hence

$$P(X + Y = 2) = P(0, 2) + P(1, 1) = e^{-\mu}p(1-p) + e^{-\mu}\mu p = e^{-\mu}p(\mu + 1 - p).$$

(b)

$$\begin{aligned} P(Y > X) &= \sum_{x<y} P(x, y) = \sum_{x=0}^{\infty}\sum_{y=x+1}^{\infty} e^{-\mu}\frac{\mu^x}{x!}p(1-p)^{y-1} \\ &= e^{-\mu}\sum_{x=0}^{\infty}\frac{\mu^x}{x!}p(1-p)^x\sum_{n=0}^{\infty}(1-p)^n \qquad (\text{set } y = x + 1 + n) \\ &= e^{-\mu}p\sum_{x=0}^{\infty}\frac{(\mu(1-p))^x}{x!}\cdot\frac{1}{1-(1-p)} \qquad (\text{geometric series}) \\ &= e^{-\mu}e^{\mu(1-p)} = e^{-\mu p}. \end{aligned}$$