Math 361/Stat 351 X1 Homework 10 Solutions Spring 2003

Graded problems: 1(d), 3(c), 4(b), 7

Problem 1.

[4.2.4] Suppose component lifetimes are exponentially distributed with mean 10 hours. Find (a) the probability that a component survives 20 hours; (b) the **median** component lifetime (see p. 165 for a definition of the median of a distribution); (c) the SD of component lifetime; (d) the probability that the average lifetime of 100 independent components exceeds 11 hours.

Solution.

(a) We need to compute $P(X \geq 20)$, where X is the life time of the component. We are given that X has exponential distribution with mean $E(X) = 10$ (measured in hours). Since for an exponential(λ) distribution $E(X) = 1/\lambda$, we must have $\lambda = 1/E(X) = 1/10$. Thus $P(X > 20) = 1 - P(X < 20) = 1 - F(20) = 1 - (1 - e^{-(1/10)\cdot 20}) = e^{-2}$, using the formula $F(x) = 1 - e^{-\lambda x}$ $(x > 0)$ for the c.d.f. of an exponential distribution.
(b) The median lifetime is the value of t for which $P(X \geq t) = 1/2 = P(X \leq t)$. Now, by the same argument as in (a), we have for $t \geq 0$ $P(X \geq t) = e^{-5/10}$, and setting this value equal to $1/2$ we get $e^{-t/10} = 1/2$ or $t = 10 \ln 2 = 6.93$.
(c) Since the lifetime X is exponentially distributed with parameter $\lambda = 1/10$, its standard deviation is, by the formula for the variance of an exponential distribution,

$$SD(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{1/\lambda^2} = 1/\lambda = 10.$$

(d) Let $X_1, X_2, \ldots, X_{100}$ be the lifetimes of the 100 independent components. The random variables X_i are i.i.d., each having exponential distribution with $\lambda = 1/10$ and $\mu = E(X_1) = 10$, $\sigma = SD(X_1) = 10$. We need to compute the probability $P(A_{100} > 11)$, where

$$A_{100} = \frac{1}{100} \sum_{i=1}^{100} X_i = \frac{1}{100} S_{100}$$

is the average lifetime (in the "naive" sense of average) of the 100 components. By normal approximation this is

$$\begin{aligned} P(A_{100} > 11) &= 1 - P(A_{100} \leq 11) = 1 - P(S_{100} \leq 1100) \\ &\approx 1 - \Phi\left(\frac{1100 - 100 \cdot 10}{\sqrt{100} \cdot 10}\right) = 1 - \Phi(1) = 0.1587. \end{aligned}$$

Problem 2.

[4.R:4] Let X be a random variable with density $f(x) = 0.5e^{-|x|}$ $(-\infty < x < \infty)$. Find (a) $P(X < 1)$, (b) $E(X)$ and $SD(X)$, (c) the c.d.f. of X^2.

Solution.

(a)
$$\begin{aligned} P(X < 1) &= \int_{-\infty}^{1} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{1} e^{-|x|}dx = \frac{1}{2}\int_{-\infty}^{0} e^{x}dx + \frac{1}{2}\int_{0}^{1} e^{-x}\,dx \\ &= \frac{1}{2}e^{x}\Big|_{-\infty}^{0} + \frac{1}{2(-1)}e^{-x}\Big|_{0}^{1} = \frac{1}{2} + \frac{1}{2}\left(-e^{-1} + 1\right) = 1 - \frac{1}{2e}. \end{aligned}$$

$$\text{(b)}\qquad E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \frac{1}{2} e^{-|x|}\,dx = 0 \quad \text{(since integrand is an odd function)}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 \frac{1}{2} e^{-|x|}\,dx = \int_{0}^{\infty} x^2 e^{-x}\,dx \quad \text{(by symmetry)}$$
$$= 2 \quad \text{(from integral table or by integration by parts)}$$

Thus, $SD(X) = \sqrt{E(X^2) - E(X)^2} = \sqrt{2 - 0^2} = \sqrt{2}$.
(c) Let $Y = X^2$. The c.d.f. of Y is defined as $F(y) = P(Y \le y)$. Since $Y = X^2 \ge 0$, we have $F(y) = 0$ for $y < 0$. On the other hand, if $y \ge 0$, then $Y \le y$ is equivalent to $-\sqrt{y} \le X \le \sqrt{y}$. Thus, for $y \ge 0$,

$$F(y) = P(Y \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} e^{-|x|}\,dx$$
$$= \int_{0}^{\sqrt{y}} e^{-x}\,dx = 1 - e^{-\sqrt{y}}.$$

Thus,

$$F(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 - e^{-\sqrt{y}} & \text{if } y \ge 0 \end{cases}$$

Problem 3.

[4.R.14] Particles arrive at a Geiger counter according to a Poisson process with rate 3 per minute. (a) Find the probability that less than 4 particles arrive in the time interval from 0 to 2 minutes. (b) Let T_n denote the arrival time, in minutes, of the nth particle. Find $P(T_1 < 1, T_2 - T_1 < 1, T_3 - T_2 < 1)$. (c) Find the conditional distribution of the number of arrivals in the time interval from 0 to 2 minutes, given that there were 10 arrivals in the time interval from 0 to 4 minutes.

Solution.

(a) Using the Poisson process notation, the probability to compute is

$$P(N(0,2) < 4) = \sum_{k=0}^{3} P(N(0,2) = k) = \sum_{k=0}^{3} e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$
$$= \sum_{k=0}^{3} e^{-6} \frac{6^k}{k!} = 61 e^{-6} = 0.15$$

$$\text{(b)}\quad P(T_1 < 1, T_2 - T_1 < 1, T_3 - T_2 < 1)$$
$$= P(W_1 < 1, W_2 < 1, W_3 < 1) \quad (\text{since } W_i = T_i - T_{i-1})$$
$$= P(W_1 < 1) P(W_2 < 1) P(W_3 < 1) \quad (\text{by the indep. of the } W_i\text{'s})$$
$$= (1 - e^{-\lambda \cdot 1})^3 = (1 - e^{-3})^3 = 0.857 \quad (\text{by the exp. distr. of the } W_i\text{'s with } \lambda = 3)$$

(c) We need to compute, for $k = 0, 1, \ldots, 10$, the conditional probability that $N(0,2) = k$ given that $N(0,4) = 10$, i.e.,

$$P(N(0,2) = k | N(0,4) = 10) = \frac{P(N(0,2) = k \text{ and } N(0,4) = 10)}{P(N(0,4) = 10)} = \frac{A}{B}$$

say. The denominator B here is

$$B = P(N(0,4) = 10) = e^{-\lambda \cdot t} \frac{(\lambda t)^{10}}{10!} = e^{-12} \frac{(12)^{10}}{10!}$$

since $N(0,4)$ has Poisson distribution with parameter $\mu = \lambda t = 3 \cdot 4 = 12$. The numerator can be evaluated (as in the example worked in class on 4/21) by rewriting the given probability (which involves the overlapping intervals $(0,2)$ and $(0,4)$) in terms of a probability involving the disjoint intervals $(0,2)$ and $(2,4)$, and applying the independence of the $N(I)$'s for disjoint intervals I:

$$\begin{aligned} A &= P(N(0,2) = k \text{ and } N(2,4) = 10-k) = P(N(0,2) = k)P(N(2,4) = 10-k) \\ &= e^{-6} \frac{(6)^k}{k!} \cdot e^{-6} \frac{(6)^{10-k}}{(10-k)!} = e^{-12} \frac{6^{10}}{k!(10-k)!}. \end{aligned}$$

Hence,

$$P(N(0,2) = k | N(0,4) = 10) = \frac{A}{B} = \frac{e^{-12} 6^{10}/(k!(10-k)!)}{e^{-12} 12^{10}/10!} = \binom{10}{k} \left(\frac{1}{2}\right)^{10}, \quad k = 0, 1, \ldots, 10.$$

This distribution can be recognized as the binomial distribution with parameters $n = 10$ and $p = 1/2$.

Problem 4.

[5.1.4] Let X and Y be independent random variables, each uniformly distributed on $(0,1)$. Find: (a) $P(|X - Y| \le 0.25)$; (b) $P(|\frac{X}{Y} - 1| \le 0.25)$; (c) $P(Y \ge X | Y \ge 0.25)$ (conditional probability).

Solution.

This problem is of the same type as a problem worked out in class on 4/23. Since X and Y are independent and uniform on $(0,1)$, the joint density is $f(x,y) = f_X(x) f_Y(y) = 1 \cdot 1 = 1$ for $0 \le x \le 1$, $0 \le y \le 1$, and the probability that (X,Y) belongs to a given region R is equal to the area of this region.
(a) Here R is the region inside the unit square $0 \le x \le 1, 0 \le y \le 1$ in which $|x - y| \le 0.25$. The area of this region is $1 - 2 \cdot (1/2)(3/4)^2 = 7/16$ (sketch the region and note that its complement consists of two right triangles of side lengths 3/4). Hence $P(|X - Y| \le 0.25) = 7/16$.
(b) Write the condition $|\frac{x}{y} - 1| \le 0.25$ as $|x - y| \le 0.25y$ or, equivalently, $0.75y \le x \le 1.25y$. The region inside the unit square represented by this condition has the form of a wedge whose complement consists of two right triangles with areas $(1/2)(4/5) = 2/5$ and $(1/2)(3/4) = 3/8$, respectively. Hence the area of this region is $1 - 2/5 - 3/8 = 9/40$, and so $P(|\frac{X}{Y} - 1| \le 0.25) = 9/40$.
(c) The conditional probability sought is $(*)$ $P(Y \ge X \text{ and } Y \ge 1/4)/P(Y \ge 1/4)$. The denominator $P(Y \ge 1/4)$ is 3/4, by the uniform distribution of Y, while the probability in the numerator is equal to the area of the region inside the unit square determined by the inequalities $y \ge x$ and $y \ge 1/4$. The region consists of the half the unit square (the part lying above the main diagonal) minus a right triangle with sides 1/4 (draw picture!). Hence its area is $1/2 - (1/2)(1/4)^2 = 15/32$, and the probability $(*)$ is $(15/32)/(3/4) = 15/24$.

Problem 5.

[5.2.4] Assume X and Y have joint density function

$$f(x,y) = 6e^{-2x-3y} \quad (x, y > 0),$$

and $f(x,y) = 0$ otherwise. Find (a) the probability $P(X \le x, Y \le y)$; (b) the marginal density of X, $f_X(x)$; (c) the marginal density of Y, $f_Y(y)$. (d) Determine (with appropriate justification) whether X and Y are independent.

Solution.

(a)
$$P(X \le x, Y \le y) = \int_{u=0}^{x}\int_{v=0}^{y} 6e^{-2u-3v}\,du dv = 6\left(\int_0^x e^{-2u}du\right)\left(\int_0^y e^{-2v}dv\right)$$
$$= 6\frac{1-e^{-2x}}{2}\cdot\frac{1-e^{-3y}}{3} = (1-e^{-2x})(1-e^{-3y}) \quad (x, y \ge 0)$$

(b)
$$f_X(x) = \int f(x,y)dy = \int_0^\infty 6e^{-2x-3y}dy$$
$$= 6e^{-2x}\int_0^\infty e^{-3y}dy = 6e^{-2x}\frac{1}{3} = 2e^{-2x} \quad (x \ge 0).$$

(c)
$$f_Y(y) = \int f(x,y)dx = \int_0^\infty 6e^{-2x-3y}dx$$
$$= 6e^{-3y}\int_0^\infty e^{-2x}dx = 6e^{-3y}\frac{1}{2} = 3e^{-3y} \quad (y \ge 0).$$

(d) X and Y are independent since, by (b) and (c), $f_X(x)f_Y(y) = (2e^{-2x})(3e^{-3y}) = 6e^{-2x-3y} = f(x,y)$ for all $x, y \ge 0$ (i.e., all x and y in the range of the joint density $f(x,y)$).
Remark: Once indepedence has been established, one can compute the probability in (a) very quickly, namely by writing it as the product of the probabilities $P(X \le x)$ and $P(Y \le y)$, which are given by $F_X(x) = 1 - e^{-2x}$ and $F_Y(y) = 1 - e^{-3y}$, respectively. However, without knowing that X and Y are independent, *and having proved so*, one has to compute this probability as a double integral.

Problem 6.

[5.2.11] Suppose X and Y are independent random variables such that X has uniform distribution on the interval $(0,1)$ and Y has exponential distribution with mean 1. Find (a) $E(X+Y)$; (b) $E(XY)$; (c) $E((X-Y)^2)$.

Solution.

First note that $E(X) = 1/2$, since X is uniform on $(0,1)$, and $E(Y) = 1$ since Y is exponential with mean 1.

(a)
$$E(X+Y) = E(X) + E(Y) = \frac{1}{2} + 1 = \frac{3}{2}$$

(b)
$$E(XY) = E(X)E(Y) = \frac{1}{2}\cdot 1 = \frac{1}{2} \quad \text{(by independence)}$$

(c) We have $E(X^2) = \int_0^1 x^2 \cdot 1dx = \frac{1}{3}$ and $E(Y^2) = \int_0^\infty y^2 e^{-y}dy = 2$ (integrate by parts twice). Thus,

$$E((X-Y)^2) = E(X^2 - 2XY + Y^2) = E(X^2) - 2E(XY) + E(Y^2) = \frac{1}{3} - 2\cdot\frac{1}{2} + 2 = \frac{4}{3}$$

Problem 7.

[5.2.21(b)] Let D denote the distance of two points picked randomly from the unit square. Do one of the following: (a) Compute $E(D^2)$ exactly. (b) Use computer simulation to obtain numerical values for $E(D^2)$ and $E(D)$.
(You only need to do one of the two parts. Part (a) is a theoretical exercise. Part (b) should be an easy programming exercise. If you choose to do the latter, attach the source code and the program output to your hw assignment.)

Solution.

Let (X_1, Y_1) and (X_2, Y_2) denote the coordinates of the two points. The X_i's and Y_i's are independent random variables, each uniformly distributed on the interval $[0, 1]$. (The independence is implied by the phrase that the points are picked "randomly" from the unit square.) The square of the distance between these two points is $D^2 = (X_2 - X_1)^2 + (Y_2 - Y_1)^2$, so

$$E(D^2) = E((X_2 - X_1)^2 + (Y_2 - Y_1)^2) = E((X_2 - X_1)^2) + E((Y_2 - Y_1)^2).$$

Squaring out and using the properties of the expectation (in particular, the fact that $E(XY) = E(X)E(Y)$ for independent X and Y), we get

$$E((X_2 - X_1)^2) = E(X_2^2) - 2E(X_1)E(X_2) + E(X_1^2) = \frac{1}{3} - 2 \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} = \frac{1}{6}$$

since $E(X_i) = \int_0^1 x dx = \frac{1}{2}$ and $E(X_i^2) = \int_0^1 x^2 dx = \frac{1}{3}$. Similarly $E((Y_2 - Y_1)^2) = \frac{1}{3}$, so $E(D^2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.