Math 361 X1 **Homework 1 Solutions** **Spring 2003**

Graded problems: 1; 2(b);3;5; each worth 3 pts., maximal score is 12 pts.

Problem 1.

A coin is tossed repeatedly. What is the probability that the second head appears at the 5th toss? (Hint: Since only the first five tosses matter, you can assume that the coin is tossed only 5 times.)

Solution.

Since only the first five tosses matter, we can take $\Omega = \{HHHHH, HHHHT, \dots\}$, i.e., the set of all $2^5 = 32$ five letter sequences of H and T, with equal probabilities. The event in question corresponds to the subset $A = \{HTTTH, THTTH, TTHTH, TTTH\}$. Since $\#(\Omega) = 32$ and $\#(A) = 4$, this event has probability $P(A) = \#(A)/\#(\Omega) = 4/2^5 = 1/8$.

Problem 2.

[1.1:8(a),(c)] Suppose two n-sided dice are rolled. Define an appropriate probability space Ω and find the probabilities of the following events. (a) the maximum of the two numbers rolled is less than or equal to 2; (b) the maximum of the two numbers rolled is exactly equal to 3. (The maximum is the larger of the two numbers; e.g., $\max(3,5) = 5$, or $\max(3,3) = 3$.)

Solution.

Letting a_1 and a_2 denote the two numbers appearing, the possible outcomes are tuples (a_1, a_2) with each a_i ranging from 1 to n. Thus, an appropriate outcome space is given by $\Omega = \{(a_1, a_2) : a_i = 1, 2, \dots, n\}$, with each outcome being equally likely.

(a) The event (a) corresponds to the subset $A = \{(a_1, a_2) : a_i = 1, 2\}$ and its probability is given by $P(A) = \dfrac{\#(A)}{\#(\Omega)} = \dfrac{2 \times 2}{n \times n} = \dfrac{4}{n^2}$ (assuming $n \geq 2$).

(b) The event (b) corresponds to the subset $B = \{(a_1, a_2) : a_i = 1, 2, \dots, n; \max(a_1, a_2) = 3\}$. If $n \leq 2$, B is obviously empty; if $n \geq 3$, B consists of the five tuples $(1,3)$, $(2,3)$, $(3,3)$, $(3,1)$, $(3,2)$, so $\#(B) = 5$. Thus, $P(B) = \dfrac{\#(B)}{\#(\Omega)} = \dfrac{5}{n^2}$ if $n \geq 3$ and $P(B) = 0$ if $n \leq 2$.

Problem 3.

Suppose each of 100 professors in a large mathematics department picks at random one of 200 courses. What is the probability that two professors pick the same course?

Solution.

This is a birthday type problem. Label the 200 courses by $1, 2, \dots, 200$, and denote the course choices of the 100 professors by $c_1, c_2, \dots, c_{100}$. An outcome can then be described by a tuple $(c_1, \dots, c_{100})$ with $c_i = 1, \dots, 200$, so an appropriate outcome space for this problem is

$$\Omega = \{(c_1, \dots, c_{100}) : c_i = 1, \dots, 200\},$$

with equally likely outcomes. The event that **no** two professors pick the same course corresponds to the subset

$$A = \{(c_1, \dots, c_{100}) : c_i = 1, \dots, 200;\ \text{all } c_i \text{ distinct}\},$$

which has probability $P(A) = \dfrac{\#(A)}{\#(\Omega)} = \dfrac{200 \times 199 \cdots \times 101}{200^{100}} \approx 6.6 \times 10^{-14}$. The given event "(at least) two professors pick the same course" is A^c and has probability $P(A^c) = 1 - P(A) \approx 1 - 6.6 \times 10^{-14}$.

Remark: The only feasible way to do this problem is by computing the probability of the complementary event, i.e., the probability that no two professors pick the same course. A direct counting of the number of cases in which (at least) one course is picked by two professors (i.e., the number of elements of A^c with A defined as above), is **exceedingly difficult**; attempting to do so with formulas such as $\binom{100}{2}100 \cdot 199^{98}$ either overcounts or undercounts the actual number of cases. To see the difficulties involved, suppose there are only 3 professors and 6 courses to choose from. Then the professors' picks can be represented by triples (a_1, a_2, a_3), where each a_i is a number between 1 and 6. The cases where two professors pick the same course fall into four types: (a, a, b), (a, b, a), (b, a, a), and (a, a, a), where b is different from a. There are $6 \cdot 5$ cases for each of the first three types, but only 6 cases for the last type. Thus, the total number of cases is $3 \cdot 6 \cdot 5 + 6$. A similar analysis with 4 professors and 8 courses would involve dozens of different types of cases, and with 200 professors the number would be in the zillions ...

Problem 4.

[1.R:12(b)] Suppose n ordinary dice are rolled. What is the chance that at least one number appears more than once?

Solution.

Another birthday type problem. Here $\Omega = \{(a_1, \dots, a_n) : a_i = 1, \dots, 6\}$; the event that all numbers are distinct is $A = \{(a_1, \dots, a_n) : a_i = 1, \dots, 6; \text{all } a_i \text{ distinct}\}$; its probability is (as before) $P(A) = \dfrac{6 \times 5 \times \cdots (6 - n + 1)}{6^n}$, and the probability that at least one number appears more than once is $1 - \dfrac{6 \times 5 \times \cdots (6 - n + 1)}{6^n}$. (Note that if $n \geq 7$, this is equal to 1, as it should be.)

Problem 5.

[1.R:16(a) (simplified)] A dormitory has 10 students, all of whom like to gossip. One of the students hears a rumor, and tells it to one of the other students picked at random. Subsequently, each student who hears the rumor tells it to a student picked at random from the dormitory *excluding himself/herself and the person from whom he/she heard the rumor*. Find the probability that the rumor is told 5 times without coming back to a student who has already heard it.

Solution.

Label the students by 1 through 10, and let s_1 be the first student to hear the rumor, s_2 the second, etc. The constraints given in italics mean that s_2 must be different from s_1 and for $i \geq 2$, s_i must be different from both s_{i-1} and s_{i-2}. An outcome may be represented by a tuple $(s_1, s_2, \dots, s_5)$ satisfying these conditions, and each of these outcomes is equally likely; thus, we can take

$$\Omega = \{(s_1, \dots, s_5) : s_i = 1, \dots, 10; s_2 \neq s_1; s_3 \neq s_1, s_2; s_4 \neq s_2, s_3; s_5 \neq s_3, s_4\}.$$

Note that $\#(\Omega) = 10 \times 9 \times 8 \times 8 \times 8$. The event that the rumor is told 5 times without coming back to a student who has already heard it corresponds to the subset

$$A = \{(s_1, \dots, s_5) : s_i = 1, \dots, 10; \text{all } s_i \text{ distinct}\}$$

with $\#(A) = 10 \times 9 \times 8 \times 7 \times 6$. Its probability is

$$P(A) = \frac{\#(A)}{\#(\Omega)} = \frac{10 \times 9 \times \cdots 6}{10 \times 9 \times 8^3} = \frac{7 \times 6}{8^2} = \frac{21}{32}.$$

Remark: If the student who first hears the rumor is not counted among the "5 times" (which is an acceptable interpretation), then $(s_1, \dots, s_5)$ in the definitions of Ω and A would have to be replaced by $(s_1, \dots, s_6)$ and the answer would be $\dfrac{10 \times 9 \times 8 \times 7 \times 6 \times 5}{10 \times 9 \times 8^4} = \dfrac{105}{256}$.