Math 408, Spring 2008
Midterm Exam 1 Solutions

1. Let A and B be events such that $P(A) = 0.3$, $P(B) = 0.6$, and $P(A|B') = 0.25$.

   (a) Find $P(A|B)$.
   Solution. We first compute $P(A \cap B)$: From the given data we get

   $$0.25 = P(A|B') = \frac{P(A \cap B')}{P(B')} = \frac{P(A) - P(A \cap B)}{1 - P(B)} = \frac{0.3 - P(A \cap B)}{1 - 0.6},$$

   so $P(A \cap B) = 0.3 - 0.25 \cdot 0.4 = \boxed{0.2}$. Hence

   $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.6} = \boxed{\frac{1}{3}}.$$

   (b) Let $C = A \cap B'$ and $D = A' \cap B$. Are the events C and D independent? Justify your answer!
   Solution. To check independence of C and D, we need to check whether $P(C \cap D)$ is equal to $P(C)P(D)$. We compute

   $$\begin{aligned} P(C) &= P(A' \cap B) = P(B) - P(A \cap B) = 0.6 - 0.2 = 0.4, \\ P(D) &= P(A \cap B') = P(A) - P(A \cap B) = 0.3 - 0.2 = 0.1, \\ P(C \cap D) &= P(A' \cap B \cap A \cap B') = P(\emptyset) = 0, \end{aligned}$$

   so $P(C)P(D) = 0.04 \neq P(C \cap D) = 0$. Hence C and D are not independent.
   Remark: The problem asked about the independence of C and D, not that of A and B' (or A' and B).

2. Suppose A, B and C are events with $P(A) = P(B) = P(C) = 1/3$, $P(A \cap B) = P(A \cap C) = P(B \cap C) = 1/4$ and $P(A \cap B \cap C) = 1/5$. Find $P(A' \cap B' \cap C')$.
   Solution. By the formula for the probability of a triple union, we have

   $$\begin{aligned} P(A \cup B \cup C) = P(A) &+ P(B) + P(C) \\ &- P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &+ P(A \cap B \cap C) \\ = 3 \cdot (1/3) &- 3 \cdot (1/4) + (1/5) = \frac{9}{20}. \end{aligned}$$

   Since $A' \cap B' \cap C' = (A \cup B \cup C)'$ (to see this, either use DeMorgan's Law, or draw a Venn diagram, we get

   $$P(A' \cap B' \cap C') = 1 - P(A \cup B \cup C) = 1 - \frac{9}{20} = \boxed{\frac{11}{20} = 0.55}.$$

3. A study of automobile accidents produced the following data:

Model year	Proportion of all vehicles	Probability of involvement in an accident
1997	0.16	0.05
1998	0.18	0.02
1999	0.20	0.03
Other	0.46	0.04

(a) If a car is selected at random, what is the probability that it was involved in an accident?

Solution. Let A denote the event that the car was involved in an accident. By the total probability rule (with "O" meaning "other")

$$\begin{aligned} P(A) &= P(A|1997)P(1997) + P(A|1998)P(1998) \\ &\quad + P(A|1999)P(1999) + P(A|O)P(O) \\ &= 0.05 \cdot 0.16 + 0.02 \cdot 0.18 + 0.03 \cdot 0.20 + 0.04 \cdot 0.46 = \boxed{0.036} \end{aligned}$$

(b) If a car was involved in an accident, what is the probability that it was from the model year 1999?

Solution. We need to compute $P(1999|A)$. By Bayes' rule and the result from part (i),

$$P(1999|A) = \frac{P(A|1999)P(1999)}{P(A)} = \frac{0.03 \cdot 0.20}{0.036} = \boxed{1/6 = 0.1666}$$

4. *(No calculators for this problem; you can leave the answer in "raw form".)* Suppose you roll two **four-faced** dice, with faces labeled $1, 2, 3, 4$, and each equally likely to appear on top. Let X denote the smaller of the two numbers that appear. (If both dice show the same number, then X is equal to that common number.)

(a) Find the probability mass function (p.m.f.) of X.

Solution. The values are $x = 1, 2, \ldots, 4$. Note that each particular pair of numbers (a, b) with $a = 1, 2, \ldots, 4, b = 1, 2, \ldots, 4$ has a $(1/4)^2 = 1/16$ probability of showing up. Now, $X = 4$ corresponds to the single outcome $(4, 4)$ of the two rolls, which has probability $1/16$, so $f(4) = 1/16$. $X = 3$ corresponds to three outcomes, $(3, 3), (3, 4), (4, 3)$, so $f(3) = 3/16$, $X = 2$ corresponds to five outcomes, $(2, 2), (2, 3), (2, 4), (4, 2), (3, 2)$, so $f(4) = 5/16$, and $X = 1$ corresponds to seven outcomes, $(1, 1), (1, 2), (1, 3), (1, 4), (4, 1), (3, 1), (2, 1)$, so $f(4) = 7/16$. Thus the distribution table is

x	1	2	3	4
$f(x)$	7/16	5/16	3/16	1/16

(b) Find $P(X < 4 \mid X > 1)$.

Solution. We have $P(X > 1) = 1 - P(X = 1) = 1 - 7/16 = 9/16$ and $P(1 < X < 4) = P(X = 2) + P(X = 3) = 5/16 + 3/16 = 1/2$, so

$$P(X < 4|X > 1) = \frac{P(X < 4 \text{ and } X > 1)}{P(X > 1)} = \frac{1/2}{9/16} = \boxed{\frac{8}{9}}.$$

(c) Find $E(2^X)$.

Solution. Using the formula $E(u(X)) = \sum_{\text{values } x} u(x)f(x)$ and the above distribution table we get

$$E(2^X) = 2^1 \cdot \frac{7}{16} + 2^2 \cdot \frac{5}{16} + 2^3 \cdot \frac{3}{16} + 2^4 \cdot \frac{1}{16}$$
$$= \frac{14 + 20 + 24 + 16}{16} = \boxed{\frac{74}{16}} \; (= 4.625)$$

5. Assume that the probability of a car insurance policyholder to have an accident in any given year is $1/4$, and that accidents in different years occur independently of each other. Suppose a customer begins a new policy at the beginning of 2008.

(a) *(No calculators for this problem; you can leave the answer in "raw form".)* Find the probability that the first accident will occur in the 5th year of the policy, i.e., in 2012.

Solution. We need $P(X = 5)$, where X denotes the year of the first accident (counting 2008 as year 1). The given assumptions imply that the occurrence of accidents can be modeled as S/F trial sequence, with a trial corresponding to a given year, and "success" meaning that an accident occurs in that year, and success probability $p = 1/4$. In this model, X is the trial of the first success, so has geometric distribution. Hence

$$P(X = 5) = (1-p)^4 p = \left(\frac{3}{4}\right)^4 \frac{1}{4} = \boxed{\frac{3^4}{4^5} = \frac{81}{1024}} \ (= 0.791).$$

(b) *(No calculators for this problem; you can leave the answer in "raw form".)* Find the probability that the first accident occurs in a leap year, i.e., in one of the years $2008, 2012, 2016, \ldots$.

Solution. The years $2008, 2012, 2016, \ldots$ represent years $1, 5, 9, \ldots$ of the policy. Thus, the probability to compute is

$$\begin{aligned} &P(X = 1 \text{ or } X = 5 \text{ or } X = 9 \cdots) \\ &\quad = P(X = 1) + P(X = 5) + P(X = 9) + \cdots \\ &\quad = (1-p)^0 p + (1-p)^4 p + (1-p)^8 p + \cdots . \end{aligned}$$

The latter expression is a geometric series and can be evaluated with the geometric series formula:

$$\begin{aligned} &= p \sum_{k=0}^{\infty} \left((1-p)^4\right)^k \\ &= \frac{p}{1-(1-p)^4} = \boxed{\frac{1/4}{1-(3/4)^4}} \ (= 0.365714), \end{aligned}$$

(c) Find the minimal value of N such that, with probability at least 98%, the policyholder will have had an accident within the first N years of the policy. *(You can use a calculator for this part.)*

Solution. The probability of the complementary event, namely that there is *no* accident within the first N years of the policy is $P(X > N) = (1-p)^N = (3/4)^N$. Thus we seek the minimal value of N such that $(3/4)^N < 0.02$. Taking logarithms and solving for N gives $N \ln(3/4) < \ln(0.02)$ or $N > |\ln(0.02)/\ln(3/4)| = 13.598$. The minimal value of N satisfying this condition is $N = \boxed{14}$.

6. *(No calculators for this problem; you can leave the answer in "raw form".)* Suppose a random variable X has moment generating function

$$M(t) = \frac{1}{8} + \frac{3}{8}e^t + \frac{3}{8}e^{2t} + \frac{1}{8}e^{3t}.$$

(a) Find the variance of X.

Solution. We use the formulas $\sigma^2 = M''(0) - M'(0)^2$. From the given formula for $M(t)$,

$$M'(t) = \frac{3}{8}e^t + \frac{6}{8}e^{2t} + \frac{3}{8}e^{3t}, \quad M'(0) = \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{3}{2},$$
$$M''(t) = \frac{3}{8}e^t + \frac{12}{8}e^{2t} + \frac{9}{8}e^{3t}. \quad M''(0) = \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = 3,$$
$$\sigma^2 = M''(0) - M'(0)^2 = 3 - \left(\frac{3}{2}\right)^2 = \boxed{\frac{3}{4}}$$

(b) Find $P(X \le 2)$.

Solution. The probability distribution of X can be read off from the coefficients of e^{xt} in the formula for $M(t)$:

$$P(X = 0) = \frac{1}{8}, \quad P(X = 1) = \frac{3}{8}, \quad P(X = 2) = \frac{3}{8}, \quad P(X = 3) = \frac{1}{8},$$

Hence

$$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \boxed{\frac{7}{8}}$$

[This is a variation of a HW problem (2.5-3 in HW 5).]

7. *(No calculators for this problem; you can leave the answer in "raw form".)* The loss X incurred by an insuranceholder has density

$$f(x) = \begin{cases} ce^{-x} & \text{if } 1 \le x < \infty, \\ 0 & \text{if } -\infty < x < 1, \end{cases}$$

where c is a constant.

(a) Find the **median** loss.

Solution. We first determine the constant c: Setting $1 = \int_{x=1}^{\infty} f(x)dx$, we get

$$1 = \int_{1}^{\infty} ce^{-x}dx = c\left[(-1)e^{-x}\right]_{1}^{\infty} = ce^{-1},$$

so $c = e^1$ and $f(x) = e^{1-x}$ for $1 \le x < \infty$. Next, we compute the c.d.f.:

$$F(x) = \int_{1}^{x} f(t)dt = \int_{1}^{x} e^{1-t}dt = \left(e^{1-1} - e^{1-x}\right) = 1 - e^{1-x} \quad (1 \le x < \infty).$$

Setting this equal to 0.5 and solving for x gives the median: $1 - e^{1-x} = 0.5$, $1 - x = \ln 0.5$, $x = \boxed{1 - \ln 0.5}$ $(= 1.693)$.

(b) Suppose that the insurance company reimburses the loss except for a deductible. At what level must the deductible be set in order for the expected insurance payment to be equal to 0.8?

Solution.

Let D denote the (unknown) deductible. Then the insurance payment is given by

$$Y = \begin{cases} 0 & \text{if } 1 \le X < D, \\ X - D & \text{if } D \le X < \infty. \end{cases}$$

We need to determine D such that $E(Y) = 0.8$. We compute

$$\begin{aligned} E(Y) &= \int_{1}^{D} 0 f(x)dx + \int_{D}^{\infty} (x - D)f(x)dx \\ &= \int_{D}^{\infty} (x - D)e^{1-x}dx \\ &= \left[(x - D)(-1)e^{1-x}\right]_{x=D}^{\infty} - \int_{D}^{\infty} (-1)e^{1-x}dx \\ &= 0 + \int_{D}^{\infty} e^{1-x}dx = \left[-e^{1-x}\right]_{D}^{\infty} = e^{1-D}. \end{aligned}$$

Setting this equal to 0.8 gives $e^{1-D} = 0.8$, $1 - D = \ln 0.8$, $\boxed{D = 1 - \ln 0.8}$ $(= 1.223)$.

[A problem of the same type appeared in Problem Set 3 (Problem 3-16) and was worked out in class.]