Math 408, Spring 2008
Midterm Exam 2 Solutions

1. (NO CALCULATORS FOR THIS PROBLEM)

   (a) Consider the following statements:

   (1) If $\operatorname{Cov}(X, Y) = 0$, then X and Y are independent.

   (2) If X and Y are independent, then $\operatorname{Cov}(X, Y) = 0$.

   Circle one of the following:

   (i) Only (1) is correct.
   (ii) Only (2) is correct.
   (iii) (1) and (2) are both correct.
   (iv) Neither (1) nor (2) are correct.

   Solution. $\boxed{\text{(ii)}}$, i.e., (2) is correct, but not (1). (Note that, while independence implies that the covariance is zero, the converse is not true. In fact, a homework problem (4.2-7) gave an example of two random variables that had 0 covariance, but were not independent.

   (b) Suppose that X and Y are independent random variables with $\operatorname{Var}(X) = 1$, $\operatorname{Var}(Y) = 2$. Find $\operatorname{Var}(1 - 2X + 3Y)$.

   Solution. (Except for a minor numerical change, this was a quiz problem.)

$$\operatorname{Var}(1 - 2X + 3Y) = 0 + (-2)^2 \operatorname{Var}(X) + 3^2 \operatorname{Var}(Y) = 4 \cdot 19 \cdot 2 = \boxed{22}.$$

   (c) Suppose X and Y are random variables such that $\operatorname{Var}(X + Y) = 9$ and $\operatorname{Var}(X - Y) = 1$. Find $\operatorname{Cov}(X, Y)$.

   Solution. We have

$$\begin{aligned}
\operatorname{Var}(X + Y) &= \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X, Y), \\
\operatorname{Var}(X - Y) &= \operatorname{Var}(X) + \operatorname{Var}(-Y) + 2\operatorname{Cov}(X, -Y) \\
&= \operatorname{Var}(X) + \operatorname{Var}(Y) - 2\operatorname{Cov}(X, Y),
\end{aligned}$$

   (since $\operatorname{Var}(-Y) = \operatorname{Var}(Y)$ and $\operatorname{Cov}(X, -Y) = -\operatorname{Cov}(X, Y)$). Subtracting these two equations and substituting the given values for $\operatorname{Var}(X + Y)$ and $\operatorname{Var}(X - Y)$, we get $9 - 1 = 4\operatorname{Cov}(X, Y)$, so $\operatorname{Cov}(X, Y) = \boxed{2}$.

2. (NO CALCULATORS FOR THIS PROBLEM) Let X be a random variable with normal distribution $N(10, 4)$, and let $Y = e^X$.

(a) Find the p.d.f. of X. (The answer should be an explicit elementary function of x, not an expression involving Φ.)

Solution. The density of a general normal distribution with parameters μ and σ is $f(x) = (1/\sqrt{2\pi}\sigma)e^{-(x-\mu)^2/(2\sigma^2)}$. Here $\mu = 10$, $\sigma = 2$, so

$$f(x) = \boxed{\frac{1}{2\sqrt{2\pi}}e^{-\frac{1}{8}(x-10)^2}, \quad -\infty < x < \infty.}$$

(b) Find the p.d.f. of Y. (Again, the answer should be an explicit elementary function of y.)

Solution. We use the distribution function technique to compute first the c.d.f. $G(y)$ and then the p.d.f $g(y)$ of Y. Since $Y = e^X$ and X ranges over the entire real axis, the possible values for Y are $0 < y < \infty$. For such y we have

$$G(y) = P(Y \le y) = P(e^X \le y) = P(X \le \ln y) = F(\ln y).$$

Differentiating and using the formula for $f(x)$ from part (a), we get

$$\begin{aligned} g(y) &= G'(y) = F'(\ln y)\frac{1}{y} = \frac{f(\ln y)}{y} \\ &= \boxed{\frac{1}{2y\sqrt{2\pi}}e^{-\frac{1}{8}(\ln y-10)^2}, \quad 0 < y < \infty.} \end{aligned}$$

3. (NO CALCULATORS FOR THIS PROBLEM) Claim amounts on a home insurance policy have density function

$$f(x) = \begin{cases} 3x^{-4} & \text{for } x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose two such claims are made, and assume the claim amounts are independent.

(a) Find the p.d.f. (density) of the *larger* of the two claims.

Solution. Let X_1 and X_2 denote the two claims, and let $X^* = \max(X_1, X_2)$ be the larger (maximum) of these claims. To find the p.d.f. $g(x)$ of X^*, we first compute the c.d.f. $G(x) = P(X^* \le x)$, using the "maximum trick": For $x \ge 1$,

$$\begin{aligned} G(x) &= P(X^* \le x) = P(\max(X_1, X_2) \le x) \\ &= P(X_1 \le x, X_2 \le x) = P(X_1 \le x)P(X_2 \le x) = F(x)^2. \end{aligned}$$

Now, $f(x) = 3x^{-4}$ for $x \ge 1$, so $F(x) = \int_1^x 3t^{-4}dt = 1 - x^{-3}$, and therefore

$$\begin{aligned} G(x) &= (1 - x^{-3})^2, \\ g(x) &= G'(x) = 2(1 - x^{-3})(-(-3)x^{-4}) \\ &= \boxed{6x^{-4} - 6x^{-7} \quad (x \ge 1)}. \end{aligned}$$

(b) Find the probability that the amount of the *second* claim is at least twice that of the *first* claim.

Solution. Let X_1 and X_2 denote, respectively, the first and second claims. Then we need to compute $P(X_2 \geq 2X_1)$. By the independence of X_1 and X_2 and the given density for an individual claim, the joint density is

$$f(x_1, x_2) = (3x_1^{-4})(3x_2^{-4}) = 9x_1^{-4}x_2^{-4}, \quad 1 < x_1, x_2 < \infty.$$

Hence

$$P(X_2 > 2X_1) = \int_{x_1=1}^{\infty}\int_{x_2=2x_1}^{\infty} 9x_1^{-4}x_2^{-4}dx_2dx_1 = \int_{x_1=1}^{\infty} 9x_1^{-4}\left[\frac{x_2^{-3}}{-3}\right]_{x_2=2x_1}^{\infty} dx_1$$

$$= \int_{x_1=1}^{\infty} 3x_1^{-4}(2x_1)^{-3}dx = \frac{3}{8}\int_{x_1=1}^{\infty} x_1^{-7}dx = \boxed{\frac{1}{16}}.$$

4. (NO CALCULATORS FOR THIS PROBLEM) Let X and Y be random variables with joint density

$$f(x, y) = \frac{3}{2}, \quad 0 \leq x \leq 1, x^2 \leq y \leq 1.$$

(a) Find the marginal density $f_Y(y)$ of Y. Be sure to specify the range.

Solution. The general formula is $f_Y(y) = \int_*^* f(x, y)dx$ with appropriate integration limits. A sketch of the region shows that the correct limits are from $x = 0$ to $x = \sqrt{y}$. Thus,

$$f_Y(y) = \int_{x=0}^{\sqrt{y}} \frac{3}{2}\,dx = \boxed{\frac{3}{2}y^{1/2}, \quad 0 \leq y \leq 1}.$$

(b) Find $E(X|1/2)$, the conditional expectation of X given that $Y = 1/2$.

Solution. The conditional density of X given $Y = 1/2$ is

$$h(x|1/2) = \frac{f(x, 1/2)}{f_Y(1/2)} = \frac{3/2}{(3/2)\sqrt{1/2}} = \boxed{\sqrt{2}, \quad 0 < x < \sqrt{1/2}}.$$

Thus

$$E(X|1/2) = \int xh(x|1/2)dx = \int_0^{\sqrt{1/2}} x\sqrt{2}dx = \sqrt{2}\frac{(1/\sqrt{2})^2}{2} = \boxed{\frac{1}{2\sqrt{2}}}.$$

5. (NO CALCULATORS FOR THIS PROBLEM) Suppose that Y is uniformly distributed on the interval $[0, 4]$ and that, given $Y = y$, X is uniformly distributed on the interval $[y, y + 2]$.

(a) Determine the joint density $f(x, y)$. Be sure to specify the range.

Solution. Since Y is uniformly distributed on $[0, 4]$, we have $f_Y(y) = 1/4$, $0 \le y \le 2$. Similarly, since, given $Y = y$, X is uniformly distributed on $[y, y+2]$, the conditional density of X given $Y = y$ is $1/2$ on the interval $[y, y + 2]$; i.e., $g(x|y) = 1/2$, $y \le x \le y + 2$ for $0 \le y \le 4$. Thus

$$f(x, y) = f_Y(y)g(x|y) = \frac{1}{4} \cdot \frac{1}{2} = \boxed{\frac{1}{8}, \quad 0 < y < 4, y < x < y + 2}$$

(b) Find the marginal density $f_X(x)$ of X. Be sure to specify the range.

Solution. [This is Problem 4.3-14 in Hogg/Tanis, from HW Assignment 10. The key is to split up the x-range into three parts, and to calculate $f_X(x)$ separately for each of the three parts.] The general formula is $f_X(x) = \int_*^* f(x, y)dy$, with appropriate integration limits. From part (a) we have $f(x, y) = 1/8$, and from a sketch we see that the correct integration limits are $0 \le y \le x$ for $0 \le x \le 2$, $x - 2 \le y \le x$ for $2 \le x \le 4$, $x - 2 \le y \le 4$ for $4 \le x \le 6$. Hence,

$$f_X(x) = \begin{cases} \int_0^x \frac{1}{8}dy = \frac{x}{8} & \text{if } 0 \le x \le 2, \\ \int_{x-2}^x \frac{1}{8}dy = \frac{1}{4} & \text{if } 2 \le x \le 4, \\ \int_{x-2}^4 \frac{1}{8}dy = \frac{6-x}{8} & \text{if } 4 \le x \le 6. \end{cases}$$

6. Assume that the amount of fluid content contained in a can of soda has normal distribution with mean 12.2 ounces and standard deviation 0.1 ounces.

(a) Find the probability that a single can of soda contains less than 12 ounces.

Solution. We need to compute $P(X < 12)$ where X is normal $N(12.2, 0.1^2)$. Standardizing, we get

$$P(X < 12) = P\left(\frac{X - 12.2}{0.1} < \frac{12 - 12.2}{0.1}\right) = P(Z < -2) = 1 - \Phi(2) = \boxed{0.0228}.$$

(b) Suppose that 50 cans are measured, and that the 50 measurements are averaged. Assuming independence, determine a value c such that, with probability 0.0228, the *average* of these 50 measurements is less than c.

Solution. We need to determine c such that $P(\overline{X} < c) = 0.9772$, where $\overline{X}$ is the sample mean of the 50 r.v.'s $X_1, \dots, X_{50}$. By the CLT, $\overline{X}$ is approximately normal with mean $\mu = 12.2$ and standard deviation $0.1/\sqrt{50} = 0.014\dots$. Standardizing, we have

$$\begin{aligned} P(\overline{X} < c) &= P\left(\frac{\overline{X} - 12.2}{0.014} < \frac{c - 12.2}{0.014}\right) \\ &= P\left(Z < \frac{c - 12.2}{0.014}\right) = \Phi\left(\frac{c - 12.2}{0.014}\right) \end{aligned}$$

Setting this equal to 0.0228, we find that $(c-12.2)/0.014 = -2$, so $c = 12.2-0.028 = \boxed{12.17}$.