

Problem 1 (Definitions and theorems)

- (i) Define a perfect number, i.e., fill in the blanks in the following sentence:
An integer $n \in \mathbf{N}$ is perfect if and only if ...

Solution: An integer $n \in \mathbf{N}$ is perfect if and only if $\sigma(n) = 2n$.

- (ii) Characterize all *even* perfect numbers in terms of their prime factorization; i.e., fill in the blanks in the following sentence:
An even integer $n \in \mathbf{N}$ is a perfect number if and only if n is of the form $n = \dots$

Solution: An even integer $n \in \mathbf{N}$ is perfect if and only if it is of the form $n = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime.

- (iii) State Carmichael's conjecture for the Euler φ -function.

Solution: Carmichael's conjecture states the following: *Given $n \in \mathbf{N}$, the equation $\varphi(x) = n$ has either no solution $x \in \mathbf{N}$, or at least two solutions.*

Problem 2 (Quadratic residues)

- (i) *Without using Euler's Criterion* (but you are free to use any other tools you know), compute the following Legendre symbols:

(a) $\left(\frac{2}{251}\right)$; (b) $\left(\frac{2}{257}\right)$; (c) $\left(\frac{3}{251}\right)$; (d) $\left(\frac{2008}{257}\right)$

(Note the different denominators in (b) and (d); this is not a misprint. Use the factorization $2008 = 2^3 \cdot 251$ and the fact that 251 and 257 are both primes. Show work. For the key steps, indicate which rule/property of the Legendre symbol you are using.)

Solution: (a) $\left(\frac{2}{251}\right) = \boxed{-1}$ (by the formula for $\left(\frac{2}{p}\right)$, since $251 \equiv 3 \pmod{8}$).

(b) $\left(\frac{2}{257}\right) = \boxed{1}$ (by the formula for $\left(\frac{2}{p}\right)$, since $257 \equiv 1 \pmod{8}$).

(c) $\left(\frac{3}{251}\right) \stackrel{\text{QR}}{=} -\left(\frac{251}{3}\right) \stackrel{\text{period.}}{=} -\left(\frac{2}{3}\right) = -(-1) = \boxed{1}$.

(d) $\left(\frac{2008}{257}\right) \stackrel{\text{mult.}}{=} \left(\frac{2}{257}\right)^3 \left(\frac{251}{257}\right) \stackrel{\text{(b), QR}}{=} 1^3 \left(\frac{257}{251}\right) \stackrel{\text{period.}}{=} \left(\frac{6}{251}\right)$,

$\left(\frac{6}{251}\right) \stackrel{\text{mult.}}{=} \left(\frac{2}{251}\right) \left(\frac{3}{251}\right) \stackrel{\text{(a),(c)}}{=} (-1)(1) = \boxed{-1}$.

- (ii) Suppose a is an integer such that $a^{50} + 1$ is divisible by 101. Determine, with explanation, whether there exists an integer b such that $b^2 - a$ is divisible by 101.

Solution: An integer b with $b^2 - a$ divisible by 101 exists if and only if the congruence $x^2 \equiv a \pmod{101}$ is solvable, i.e., if and only if $\left(\frac{a}{101}\right) = 1$. By Euler's criterion, $\left(\frac{a}{101}\right) \equiv a^{(101-1)/2} = a^{50} \pmod{101}$. Since $a^{50} + 1$ is divisible by 101, we have $a^{50} \equiv -1 \pmod{101}$. Hence $\left(\frac{a}{101}\right) = -1$, so no such integer b exists.

Problem 3 (Primitive roots and orders)

- (i) For each of the following moduli find the number of incongruent primitive roots to this modulus. If there are no primitive roots, explain why. (Note that 251 is a prime.)
(a) 251, (b) 502, (c) 2008.

Solution: (a) 251 is an odd prime, so a primitive root exists. The number of incongruent primitive roots is $\varphi(\varphi(251)) = \varphi(250) = \varphi(5^3 \cdot 2^1) = 5^2(5-1)(2-1) = \boxed{100}$.

(b) 502 is of the form $2p$, where $p = 251$ is an odd prime, so a primitive root exists. The number of incongruent primitive roots is $\varphi(\varphi(502)) = \varphi(2 \cdot 251) = \varphi(250) = \varphi(5^3 \cdot 2^1) = 5^2(5-1)(2-1) = \boxed{100}$.

(c) 2008 has $\boxed{\text{no primitive root}}$, since 2008 is of the form $2^\alpha \cdot p$, with $\alpha \geq 2$ and p an odd prime.

(ii) It is known that 2 is a primitive root modulo 101. (Note that 101 is prime.) Find:

(a) The total number of incongruent integers with order exactly 25 modulo 101.

(b) An integer which has order exactly 25 modulo 101. (Just find *one* such integer; express it as an integer a in the range $1 \leq a \leq 101$, and not, for example, as a power of 2.)

(c) $2^{50} \pmod{101}$. (Express your answer as an integer in the range $1 \leq a \leq 101$.)

Solution: (a) In general, if p is an odd prime, then for any positive divisor $d \mid \varphi(p) = p-1$ there are exactly $\varphi(d)$ incongruent integers with order d . Here, $p = 101$, $d = 25$, and $\varphi(25) = 5 \cdot 4 = \boxed{20}$, so there are 20 incongruent integers with order 25 modulo 101.

(b) Since 2 has order $\varphi(101) = 100$, we have, by the order formula, for any positive integer i , $\text{ord}_{101} 2^i = \text{ord}_{101} 2 / (\text{ord}_{101} 2, i) = 100 / (100, i)$. Thus, 2^i has order 25 if and only if $(100, i) = 4$. One such integer is $2^4 = \boxed{16}$.

(c) Since 2 is a primitive root, 2 has order $\varphi(101) = 100$, so, modulo 101, we have $(2^{50})^2 = 2^{100} \equiv 1$, but $2^{50} \not\equiv 1$. Hence $2^{50} \equiv -1 \equiv \boxed{100} \pmod{101}$.

Problem 4 (Arithmetic functions)

(i) Find the following values of arithmetic functions. (Show work or give brief explanation in each case.) You can leave answers in “unmultiplied” form, such as $3^2 \cdot 251$. Note that 2008 has prime factorization $2^3 \cdot 251$.

(a) $\mu(2008)$; (b) $\nu(2008)$; (c) $\sigma(2008)$.

Solution: (a) $\mu(2^3 \cdot 251^1) = \mu(2^3)\mu(251) = 0(-1) = \boxed{0}$.

(b) $\nu(2^3 \cdot 251^1) = (3+1)(1+1) = 4 \cdot 2 = \boxed{8}$.

(c) $\sigma(2^3 \cdot 251^1) = (2^4 - 1)(251 + 1) = \boxed{15 \cdot 252} (= 3780)$.

(ii) Find all positive integers n with exactly 101 positive divisors, or show that no such integers exist. (Note that 101 is prime.)

Solution: Writing n in the standard prime factorization $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ with distinct primes p_i and $\alpha_i \in \mathbf{N}$, the number of divisors of n is given by $\nu(n) = (\alpha_1 + 1) \dots (\alpha_r + 1)$, so $\nu(n) = 101$ if and only if $(\alpha_1 + 1) \dots (\alpha_r + 1) = 101$. Since 101 is prime, this holds if and only if $r = 1$ and $\alpha_1 = 100$. Thus, the integers n with $\nu(n) = 101$ are exactly those of the form $\boxed{n = p^{100}}$ with p an arbitrary prime.

(iii) Evaluate $(\mu \star \nu)(10!)$.

Solution: Let $f = \mu \star \nu$. Since $\nu = 1 \star 1$, and μ is the Dirichlet inverse of the function 1, we have $f = \mu \star (1 \star 1) = (\mu \star 1) \star 1 = 1$. Thus, $f(n) = 1$ for all $n \in \mathbf{N}$, and, in particular, $f(10!) = \boxed{1}$.