

Chapter 6

Continued fractions

6.1 Definitions and notations

Definition (Continued fractions) A finite or infinite expression of the form

$$(6.1) \quad a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}},$$

where the a_i are real numbers, with $a_1, a_2, \dots > 0$, is called a **continued fraction** (c.f.). The numbers a_i are called the **partial quotients** of the c.f.

The continued fraction (6.1) is called **simple** if the partial quotients a_i are all integers. It is called **finite** if it terminates, i.e., if it is of the form

$$(6.2) \quad a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n}}},$$

and **infinite** otherwise.

Notation (Bracket notation for continued fractions) The continued fractions (6.1) and (6.2) are denoted by $[a_0, a_1, a_2, \dots]$ and $[a_0, a_1, a_2, \dots, a_n]$, respectively. In particular,

$$[a_0] = a_0, \quad [a_0, a_1] = a_0 + \frac{1}{a_1}, \quad [a_0, a_1, a_2] = a_0 + \frac{1}{a_1 + \frac{1}{a_2}}, \quad \dots$$

Remarks (i) Note that the first term, a_0 , is allowed to be negative or 0, but all subsequent terms a_i must be positive. This requirement ensures that there are no zero denominators and that any finite c.f. (6.2), and all of its convergents, are well-defined.

(ii) In the sequel we will almost exclusively focus on the case of simple c.f.'s, i.e., c.f.'s where all partial quotients are integers. There one important exception are c.f.'s of the

form $[a_0, a_1, \dots, a_n, x]$, where a_0, a_1, \dots, a_n are integers (with a_i positive for $i \geq 1$), but the final partial quotient, x , can be any positive real number.

Definition (Convergents) Let $\alpha = [a_0, a_1, \dots]$ be a finite or infinite c.f. Then the i -th convergent of α is defined as the c.f.

$$C_i = [a_0, a_1, \dots, a_i].$$

Definition (Convergence of infinite continued fractions) An infinite c.f. $[a_0, a_1, \dots]$ is called **convergent** if its sequence of convergents converges in the usual sense, i.e., if the limit $\lim_{i \rightarrow \infty} C_i$ exists and is equal to some real number α . In this case, we say that the continued fraction $[a_0, a_1, a_2, \dots]$ **represents** the number α , or is a **continued fraction expansion** of α , and we write

$$\alpha = [a_0, a_1, a_2, \dots].$$

Theorem 6.1 (Convergence of infinite simple c.f.'s) *Any infinite simple c.f. $[a_0, a_1, \dots]$ is convergent.*

6.2 Expansions of real numbers into continued fractions

Proposition 6.2 (Continued fraction algorithm) *Given a real number α , define successively real numbers $\alpha_0, \alpha_1, \dots$, and integers a_0, a_1, \dots by*

$$\begin{aligned} \alpha_0 &= \alpha, & a_0 &= [\alpha_0], \\ \alpha_1 &= \frac{1}{\alpha_0 - [\alpha_0]}, & a_1 &= [\alpha_1], \\ \alpha_2 &= \frac{1}{\alpha_1 - [\alpha_1]}, & a_2 &= [\alpha_2], \\ &\dots & &\dots \end{aligned}$$

where $[x]$ denotes the integer part of x (i.e., the “floor function”). Stop the algorithm if α_n is an integer (and thus $a_n = \alpha_n$); otherwise continue indefinitely. Then $[a_0, a_1, \dots]$ is a simple c.f. that represents the number α . Moreover, for any $i \geq 0$ we have

$$\alpha_i = [a_i, a_{i+1}, \dots], \quad \alpha = [a_0, a_1, \dots, a_{i-1}, \alpha_i].$$

Theorem 6.3 (Continued fraction expansion of rational numbers) *Any finite simple c.f. represents a rational number. Conversely, any rational number α can be expressed as a simple finite c.f. $\alpha = [a_0, a_1, \dots, a_n]$. Moreover, under the requirement that $a_n > 1$, this representation is unique. Thus, there is a one-to-one correspondence between rational numbers and finite simple c.f.'s with last partial quotient greater than 1.*

Theorem 6.4 (Continued fraction expansion of irrational numbers) *Any infinite simple c.f. represents an irrational number. Conversely, any irrational number α can be expressed as a simple infinite c.f. $\alpha = [a_0, a_1, a_2, \dots]$, and this representation is unique. Thus, there is a one-to-one correspondence between irrational numbers and infinite simple c.f.'s.*

Theorem 6.5 (Continued fraction expansion of quadratic irrationals) *The c.f. expansion of a quadratic irrational (i.e., a solution of a quadratic equation with integer coefficients) is eventually periodic, i.e., of the form*

$$[a_0, \dots, a_N, \overline{a_{N+1}, \dots, a_{N+p}}],$$

where the bar indicates the periodic part. Conversely, any infinite simple c.f. that is eventually periodic represents a quadratic irrational. Thus, there is a one-to-one correspondence between quadratic irrationals and infinite, eventually periodic simple c.f.'s.

6.3 Convergents

Proposition 6.6 (Algorithm for convergents) *Let $\alpha = [a_0, a_1, \dots]$ be a simple c.f. The i -th convergent to α , $C_i = [a_0, a_1, \dots, a_i]$, is given by $C_i = p_i/q_i$ for $i \geq 0$, where p_i and q_i are integers defined recursively by*

$$(6.3) \quad \begin{cases} p_{-1} = 1, & q_{-1} = 0, \\ p_0 = a_0, & q_0 = 1, \\ p_i = a_i p_{i-1} + p_{i-2}, & q_i = a_i q_{i-1} + q_{i-2} \quad (i \geq 1). \end{cases}$$

Remarks (i) Note that the values $(p_{-1}, q_{-1}) = (1, 0)$ have no natural interpretation in terms of convergents; there is no convergent C_{-1} corresponding to index $i = -1$, and the fraction p_{-1}/q_{-1} is not defined. However, these values are needed (along with the values $(p_0, q_0) = (a_0, 1)$) to get the recursive definition started.

(ii) The equations for q_i are the same as those for p_i , except for different initial values: the p_i 's start out with 1 and a_0 at $i = -1, 0$, while the q_i 's start out with 0 and 1.

Theorem 6.7 (Properties of convergents) *The convergents $C_i = p_i/q_i$, (with p_i and q_i defined by (6.3)) of an infinite simple continued fraction $\alpha = [a_0, a_1, a_2, \dots]$ satisfy:*

- (i) $(p_i, q_i) = 1$ for $i = 0, 1, \dots$; i.e., the fractions p_i/q_i are reduced.
- (ii) $q_1 < q_2 < \dots$; i.e., for $i \geq 1$, the denominators q_i are strictly increasing.
- (iii) $C_0 < C_2 < C_4 < \dots < \alpha < \dots < C_5 < C_3 < C_1$. That is, the even-indexed convergents form an increasing sequence, while the odd-indexed convergents form a decreasing sequence, with the value of the c.f. sandwiched between both sequences.
- (iv) $|C_i - C_{i+1}| = 1/(q_i q_{i+1})$ for $i = 0, 1, 2, \dots$. That is, the difference between two consecutive convergents is equal to the reciprocal of the product of the two denominators.

6.4 Rational approximations

Theorem 6.8 (Approximation of irrational numbers by rationals) *Let α be an irrational number, and let p_i/q_i be the convergents to the simple c.f. fraction expansion of α .*

(i) Any convergent p_i/q_i satisfies

$$\left| \frac{p_i}{q_i} - \alpha \right| < \frac{1}{q_i q_{i+1}} < \frac{1}{q_i^2}.$$

(ii) Conversely, any rational number a/b with $a \in \mathbf{Z}$, $b \in \mathbf{N}$, $(a, b) = 1$, that satisfies

$$\left| \frac{a}{b} - \alpha \right| < \frac{1}{2b^2},$$

is a convergent to α , i.e., $a/b = p_i/q_i$ for some i .

(iii) Among all rational numbers with denominator $\leq q_i$, the convergent p_i/q_i is the best-possible approximation to α ; i.e., for any rational number a/b with $a \in \mathbf{Z}$, $b \in \mathbf{N}$, and $1 \leq b \leq q_i$,

$$\left| \frac{p_i}{q_i} - \alpha \right| \leq \left| \frac{a}{b} - \alpha \right|,$$

with equality if and only if $a/b = p_i/q_i$.