Math 531 (Analytic Number Theory)
Fall 2005

Exam 1
Solutions

Problem 1

Let $f(n) = \sigma(n)/n$, where $\sigma(n)$ is the sum of divisors function.

(i) Determine a function g such that $f = 1 * g$.

Solution. Since $\sigma = 1 * \mathrm{id}$, we have $f = \sigma/\mathrm{id} = (1 * \mathrm{id})/\mathrm{id} = 1/\mathrm{id} * 1$, so $g = 1/\mathrm{id}$.

(ii) Using (i), obtain an estimate for the sum $S(x) = \sum_{n \le x} f(n)$ with error term $O(\log x)$. (Evaluate any constants appearing in this estimate.)

Solution. By the identity of (i) we have $f(n) = \sum_{d|n} 1/d$, so

$$\begin{aligned}
\sum_{n\le x} f(n) &= \sum_{d \le x} \frac{1}{d} \sum_{n \le x/d} 1 = \sum_{d \le x} \frac{[x/d]}{d} \\
&= x \sum_{d \le x} \frac{1}{d^2} + O\left(\sum_{d \le x} \frac{1}{d} \right) \\
&= x \sum_{d=1}^{\infty} \frac{1}{d^2} + O\left(x \sum_{d > x} \frac{1}{d^2} \right) + O\left(\sum_{d \le x} \frac{1}{d} \right) \\
&= \frac{\pi^2}{6} x + O(1) + O(\log x) = \frac{\pi^2}{6} x + O(\log x),
\end{aligned}$$

for $x \ge 2$, say.

Problem 2

Let α be a fixed non-zero real number, and let $S_\alpha(x) = \sum_{p \le x} p^{-1-i\alpha}$. Use the prime number theorem in the form $\pi(x) = x/\log x + O(x/\log^2 x)$ to derive an estimate for $S_\alpha(x)$ with error term $O_\alpha(1/\log x)$. (The estimate should not involve an integral; you can leave constants appearing in this estimate unspecified.)

Solution. By partial summation,

$$S_\alpha(x) = \pi(x)x^{-1-i\alpha} - (-1-i\alpha)I(x),$$

where

$$I(x) = \int_2^x \pi(t)t^{-2-i\alpha}dt.$$

The first term here is $O(1/\log x)$ by Chebyshev's estimate, so it remains to estimate the integral $I(x)$. Setting $\pi(x) = (x/\log x)(1+\delta(x))$, the integral splits into $I_1(x) + I_2(x)$, where

$$I_1(x) = \int_2^x \frac{1}{t^{1+i\alpha}\log t}dt, \quad I_2(x) = \int_2^x \frac{\delta(t)}{t^{1+i\alpha}\log t}dt.$$

Since, by hypothesis, $\delta(t) = O(1/\log t)$ and $\int_2^\infty t^{-1}(\log t)^{-2}dt = \int_{\log 2}^\infty u^{-2}du < \infty$, the integral $I_2(x)$ converges absolutely when extended to infinity, and we have

$$I_2(x) = C_1 + O\left(\int_x^\infty t^{-1}(\log t)^{-2}dt\right) = C_1 + O\left(\frac{1}{\log x}\right),$$

where $C_1 = C_1(\alpha)$ is a constant (depending on α). The integral $I_1(x)$ cannot be evaluated in terms of elementary functions, but we can use integration by parts to obtain an estimate for it with the desired accuracy:

$$\begin{aligned} I_1(x) &= \left.\frac{t^{-i\alpha}}{-i\alpha\log t}\right|_2^x - \int_2^x \frac{1}{i\alpha t^{1+i\alpha}(\log t)^2}dt \\ &= -\frac{x^{-i\alpha}}{i\alpha\log x} + \frac{2^{-i\alpha}}{i\alpha\log 2} - \frac{1}{i\alpha}I_3(x), \end{aligned}$$

say. The integral $I_3(x)$ here is of the same type as $I_2(x)$ (with $1/\log t$ in place of the function $\delta(t)$), so we have $I_3(x) = C_2 + O_\alpha(1/\log x)$, where $C_2 = C_2(\alpha)$ is a constant. Altogether we have

$$\begin{aligned} S_\alpha(x) &= (1+i\alpha)\left(C_1(\alpha) + \frac{2^{-i\alpha}}{i\alpha\log 2} - \frac{1}{i\alpha}C_2(\alpha)\right) + O_\alpha\left(\frac{1}{\log x}\right) \\ &= C(\alpha) + O_\alpha\left(\frac{1}{\log x}\right), \end{aligned}$$

where $C(\alpha)$ is a constant. This is an estimate of the desired form. In particular, it shows that, when $\alpha \neq 0$, the infinite series $\sum_p p^{-1-i\alpha}$ converges, with sum $C(\alpha)$.

Problem 3

(Quickies) No proofs expected here; just state the required statements or formulas, indicating briefly how you arrived at these (e.g., which formula/result you were using).

(i) Evaluate the function $f(n) = \sum_{d|n} 2^{\omega(d)}\mu(n/d)$, where $\omega(d)$ is the number of distinct prime factors of d.

Solution. Note that the function f is the convolution of the functions 2^ω and μ. Since μ and 2^ω are multiplicative (the latter since it is the exponential of an additive function), so is f, so it suffices to examine f at prime powers. We have $f(p^m) = (2^\omega * \mu)(p^m) = 2^{\omega(p^m)} - 2^{\omega(p^{m-1})}$, which reduces to $2-2=0$ if $m \geq 2$, and to $2-1=1$ if $m=1$. Thus, f is the characteristic function of the squarefree integers, i.e., $f = \mu^2$.

(ii) Let $f(n) = \sum_{d|n!} \Lambda(d)$. State an asymptotic estimate for $f(n)$ with suitable error term ($O(\log n)$ is enough). (Note the factorial in $n!$)

Solution. Since $\Lambda * 1 = \log$, $f(n) = \log n! = \sum_{k=1}^n \log k$, so by Stirling's formula,

$$f(n) = n(\log n - 1) + O(\log n).$$

(iii) Let $n_k = \prod_{i=1}^k p_i$, where p_i is the i-th prime. State an asymptotic formula for $\phi(n_k)/n_k$, as $k \to \infty$, i.e., find a simple smooth function $f(k)$ such that $\phi(n_k)/n_k \sim f(k)$ as $k \to \infty$.

Solution. Since ϕ is multiplicative and $\phi(p^m) = p^m(1-1/p)$ for any prime power p^m, we have

$$\frac{\phi(n_k)}{n_k} = \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right) \sim \frac{e^{-\gamma}}{\log p_k} \sim \frac{e^{-\gamma}}{\log k}$$

by Mertens' formula and the estimate (which follows from the PNT) $\log p_k = \log((1+o(1))k\log k) = (1+o(1))\log k$.

(iv) Given $x \geq 1$ let $R(x)$ denote the number of pairs (n,m) of positive integers whose product nm is $\leq x$. State an estimate for $R(x)$ with suitable error term.

Solution. For fixed h, the number of pairs (n,m) with $nm = h$ is $d(h)$, so $R(x)$ is equal to the divisor sum $\sum_{h\leq x} d(h)$, and Dirichlet's theorem gives

$$R(x) = x\log x + (2\gamma - 1)x + O(\sqrt{x}).$$