Math 531 (Analytic Number Theory)
Fall 2005

Exam 2
Solutions

Problem 1

Let f be the multiplicative function defined on prime powers by

$$f(p^m) = \begin{cases} 2 & \text{if } m = 1, \\ 1 & \text{if } m = 2, \\ 0 & \text{if } m > 2. \end{cases}$$

(i) Express the Dirichlet series of f, $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$, in terms of the Riemann zeta function.

(ii) Express f as a Dirichlet convolution $f = g * h$, where g and h are well-known arithmetic functions.

Solution. (i) Expanding $F(s)$ into an Euler product, we get, for $\sigma > 1$,

$$\begin{aligned} F(s) &= \prod_p \left(1 + \frac{2}{p^s} + \frac{1}{p^{2s}}\right) = \prod_p \left(1 + \frac{1}{p^s}\right)^2 \\ &= \prod_p \left(1 - \frac{1}{p^{2s}}\right)^2 \left(1 - \frac{1}{p^s}\right)^{-2} = \left(\frac{\zeta(s)}{\zeta(2s)}\right)^2. \end{aligned}$$

(ii) From the above calculation we have $F(s) = G(s)^2$, where

$$G(s) = \prod_p \left(1 + \frac{1}{p^s}\right) = \prod_p \left(1 + \sum_{m=1}^{\infty} \frac{\mu^2(p^m)}{p^{ms}}\right) = \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n^s}.$$

Hence $f = \mu^2 * \mu^2$, i.e., f is the convolution of the characteristic function of the squarefree integers with itself.

Problem 2

Let a and q be positive integers with $(a,q)=1$. Express the Dirichlet series $\sum_{n\equiv a \bmod q}\mu(n)n^{-s}$ in the half-plane $\sigma>1$ in terms of Dirichlet L-functions. (Hint: An Euler product expansion may be helpful at a certain point in the argument.)

Solution. Using the orthogonality relation to "eliminate" the summation condition $n\equiv a \bmod q$, we have, for $\sigma>1$,

$$\begin{aligned}\sum_{n\equiv a \bmod q}\frac{\mu(n)}{n^s}&=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\frac{1}{\phi(q)}\sum_{\chi \bmod q}\overline{\chi(a)}\chi(n)\\&=\frac{1}{\phi(q)}\sum_{\chi \bmod q}\overline{\chi(a)}\sum_{n=1}^{\infty}\frac{\mu(n)\chi(n)}{n^s}.\end{aligned}$$

The inner sum here is the Dirichlet series for the function $f=\mu\chi$. Expanding this series into an Euler product, we see that

$$\sum_{n=1}^{\infty}\frac{\mu(n)\chi(n)}{n^s}=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)=L(s,\chi)^{-1}.$$

Hence

$$\sum_{n\equiv a \bmod q}\frac{\mu(n)}{n^s}=\frac{1}{\phi(q)}\sum_{\chi \bmod q}\overline{\chi(a)}L(s,\chi)^{-1}.$$

Problem 3

Let $f(n)$ denote the characteristic function of the prime powers, and let $F(s)=\sum_{n=1}^{\infty}f(n)n^{-s}$ be the Dirichlet series of f. Show that, for $\sigma>1$, $F(s)=\log\zeta(s)+G(s)$, where log denotes the principal branch of the logarithm, and $G(s)$ is a function that is analytic in the half-plane $\sigma>1/2$.

Solution. For $\sigma>1$ we have

$$\begin{aligned}\log\zeta(s)&=-\sum_p\log\left(1-\frac{1}{p^s}\right)\\&=\sum_p\sum_{m=1}^{\infty}\frac{1}{mp^{ms}}\\&=\sum_p\sum_{m=1}^{\infty}\frac{1}{p^{ms}}-\sum_p\sum_{m=2}^{\infty}\frac{1-1/m}{p^{ms}}\\&=F(s)-G(s),\end{aligned}$$

where $G(s)=\sum_{n=1}^{\infty}g(n)n^{-s}$ with

$$g(n)=\begin{cases}1-1/m & \text{if } n=p^m,\ m\ge 2,\\ 0 & \text{otherwise.}\end{cases}$$

Since

$$\sum_{n=1}^{\infty} \frac{|g(n)|}{n^{\sigma}} \le \sum_p \sum_{m=2}^{\infty} \frac{1}{p^{m\sigma}} \ll \sum_p \frac{1}{p^{2\sigma}} < \infty$$

for $\sigma > 1/2$, the Dirichlet series $G(s)$ converges absolutely in this half-plane and therefore represents an analytic function there.

Problem 4

Let α be a fixed real number, and for real non-integral $x \ge 2$, let $S(x,\alpha) = \sum_{n \le x} \Lambda(n) n^{i\alpha}$.

(i) Express $S(x,\alpha)$ as a complex integral involving the Riemann zeta function. (Just state the formula. No proof is required, and you can ignore questions of convergence.)

(ii) Using this representation, give a *heuristic* (i.e., informal) derivation of the main term of an asymptotic formula for $S(x,\alpha)$. (Again, proofs or rigorous arguments are not expected, but you should explain how the main term in your formula arises from the integral representation.)

Solution. (i) Let α be fixed and $f(n) = \Lambda(n) n^{i\alpha}$. Then f has Dirichlet series

$$\begin{aligned} F(s) &= \sum_{n=1}^{\infty} \frac{\Lambda(n) n^{i\alpha}}{n^s} \\ &= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{s-i\alpha}} = -\frac{\zeta'}{\zeta}(s - i\alpha), \end{aligned}$$

and this series converges absolutely in $\sigma > 1$. Hence by Perron's formula, we have, for $c > 1$ and any non-integral $x \ge 2$,

$$S(x,\alpha) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'}{\zeta}(s - i\alpha) \right) \frac{x^s}{s} ds,$$

where $\int_{c-i\infty}^{c+i\infty}$ is to be interpreted as a symmetric limit.

(ii) Moving the path of integration to the left of the line $\sigma = 1$, we pick up a contribution from the pole of $(\zeta'/\zeta)(s - i\alpha)$ at $s = 1 + i\alpha$ (corresponding to the pole of the zeta function at 1): This contribution is equal to

$$\operatorname{Res}\left(-\frac{\zeta'}{\zeta}(s - i\alpha)\frac{x^s}{s}, s = 1 + i\alpha \right) = \frac{x^{1+i\alpha}}{1 + i\alpha}.$$

Provided the new path is within the zero-free region of the function $\zeta(s - i\alpha)$, there are no contributions from other singularities of the integrand, so the above term is the expected main term in an asymptotic estimate for $S(x,\alpha)$, i.e., we expect an estimate of the form

$$S(x,\alpha) \approx \frac{x^{1+i\alpha}}{1 + i\alpha}.$$