

Math 531 (Analytic Number Theory)
Fall 2005

Final Exam
Solutions

Problem 1

Given a positive integer k , let A_k denote the set of positive integers that are composed only of prime factors of k ; for example, $A_{12} = A_6 = \{2^m 3^n : m, n \in \mathbb{N} \cup \{0\}\}$.

(i) Show that the series $\sum_{n \in A_k} n^{-1}$ converges, with sum $S(k)$, say, and express $S(k)$ in terms of familiar arithmetic functions.

(ii) Let $N_k(x) = \#\{n \leq x : n \in A_k\}$. Determine the order of magnitude of $N_6(x)$, i.e., find a simple smooth function $\psi(x)$ such that $N_6(x) \asymp \psi(x)$ for sufficiently large x .

Solution. (i) Let $f_k(n)$ denote the characteristic function of A_k . It is easy to see that f_k is multiplicative, with $f_k(p^m) = 1$ if $p|k$, and $f_k(p^m) = 0$ if $p \nmid k$. Thus, expanding the Dirichlet series of f_k into an Euler product, we obtain

$$\begin{aligned} \sum_{n \in A_k} \frac{1}{n} &= \sum_{n=1}^{\infty} \frac{f_k(n)}{n} = \prod_{p|k} \left(1 + \sum_{m=1}^{\infty} \frac{1}{p^m} \right) \\ &= \prod_{p|k} \left(1 - \frac{1}{p} \right)^{-1} = \frac{k}{\phi(k)}. \end{aligned}$$

Thus, the series $\sum_{n \in A_k} 1/n$ converges, with sum $S(k) = k/\phi(k)$.

(ii) We claim that $N_6(x) \asymp (\log x)^2$. The upper bound follows from the estimate

$$\begin{aligned} N_6(x) &= \#\{(m, n) \in \mathbb{N} \cup \{0\} : m \log 2 + n \log 3 \leq \log x\} \\ &\leq \#\{(m, n) \in \mathbb{N} \cup \{0\} : 0 \leq m, n \leq \log x / \log 2\} \ll (\log x)^2, \end{aligned}$$

and the lower bound follows from

$$\begin{aligned} N_6(x) &= \{(m, n) \in \mathbb{N} \cup \{0\} : m \log 2 + n \log 3 \leq \log x\} \\ &\geq \{(m, n) \in \mathbb{N} \cup \{0\} : 0 \leq m, n \leq \log x / (2 \log 3)\} \gg (\log x)^2. \end{aligned}$$

Problem 2

Define an arithmetic function f by $f(1) = 1$ and $f(n) = 2^{r(n)}$ for $n \geq 2$, where $r(n) = \#\{p|n : p^2|n\}$ is the number of prime divisors of n whose square also divides n . (Thus, for example, $f(531) = f(3^2 \cdot 59) = 2^1 = 2$, while $f(505) = f(5 \cdot 101) = 2^0 = 1$.) Obtain an asymptotic estimate for the summatory function $S(x) = \sum_{n \leq x} f(n)$, with as good an error term as you can get. (An estimate with error $O(\sqrt{x})$ will earn full credit. Partial credit may be given for a weaker estimate.)

Solution. From the definition it is clear that the function f is multiplicative, with $f(p^m) = 2$ if $m \geq 2$ and $f(p^m) = 1$ if $m = 1$. Let $g = f * \mu$. Since f is multiplicative, so is g . On prime powers p^m , we have $g(p^m) = f(p^m) - f(p^{m-1}) = 1$ if $m = 2$ and $g(p^m) = 0$ if $m = 1$ or $m \geq 3$. Thus, $g(n)$ is the characteristic function of squares of squarefree numbers, i.e., $g(n) = \mu^2(m)$ if $n = m^2$, and $g(n) = 0$ if n is not a square. It follows that, for $x \geq 1$,

$$\begin{aligned} \sum_{n \leq x} f(n) &= \sum_{m \leq \sqrt{x}} \mu^2(m) \left[\frac{x}{m^2} \right] \\ &= x \sum_{m \leq \sqrt{x}} \frac{\mu^2(m)}{m^2} + O\left(\sum_{m \leq \sqrt{x}} 1 \right) \\ &= x \sum_{m=1}^{\infty} \frac{\mu^2(m)}{m^2} + O\left(\sum_{m > \sqrt{x}} \frac{1}{m^2} \right) + O(\sqrt{x}) \\ &= x \frac{\zeta(2)}{\zeta(4)} + O(\sqrt{x}) = \frac{\zeta(2)}{\zeta(4)} x + O(\sqrt{x}), \end{aligned}$$

where in the last step we used

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n^s} &= \prod_p \left(1 + \frac{1}{p^s} \right) \\ &= \prod_p \left(1 - \frac{1}{p^s} \right)^{-1} \prod_p \left(1 - \frac{1}{p^{2s}} \right) = \frac{\zeta(s)}{\zeta(2s)} \quad (\sigma > 1). \end{aligned}$$

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Problem 3

Let $f(x)$ be integrable on each finite interval $[1, x]$ ($x \geq 1$) and satisfying $f(x) \sim Ax^\alpha$ as $x \rightarrow \infty$ for some constants $A \neq 0$ and $\alpha \geq 0$. Let $F(s) = \int_1^\infty f(x)x^{-s}dx$. Using a careful ϵ - x_0 argument obtain an asymptotic formula for $F(s)$ as $s \rightarrow 1 + \alpha +$ (i.e., for s tending to $\alpha + 1$ from the right through real values).

Solution. We will show that

$$F(s) \sim \frac{A}{s - \alpha - 1} \quad (s \rightarrow 1 + \alpha+).$$

or, equivalently,

$$(1) \quad \lim_{s \rightarrow 1 + \alpha+} F(s)(s - \alpha - 1) = A.$$

Let $\epsilon > 0$ be given. By the hypothesis $f(x) \sim Ax^\alpha$ there exists an $x_0 = x_0(\epsilon) \geq 1$ such that

$$(2) \quad |f(x) - Ax^\alpha| \leq \epsilon x^\alpha \quad (x \geq x_0).$$

It follows that, for $s > \alpha + 1$,

$$\begin{aligned} \left| F(s) - \frac{A}{s - \alpha - 1} \right| &= \left| \int_1^\infty (f(x) - Ax^\alpha)x^{-s}dx \right| \\ &\leq \int_1^{x_0} |f(x) - Ax^\alpha|dx + \int_{x_0}^\infty \epsilon x^{\alpha-s}dx \\ &\leq C(\epsilon) + \frac{\epsilon}{s - \alpha - 1}, \end{aligned}$$

where $C(\epsilon)$ is a constant depending only on s , but not on s . Hence

$$\limsup_{s \rightarrow 1 + \alpha+} |F(s)(s - \alpha - 1) - A| \leq \epsilon,$$

and since ϵ was arbitrary, we obtain (1).

Problem 4

Let

$$S(x) = \sum_{p \leq x} \frac{\log p}{\sqrt{p}}.$$

State an estimate for $S(x)$ that is equivalent to the estimate

$$(1) \quad \pi(x) = \frac{x}{\log x} \left(1 + O\left(\frac{1}{\log x}\right) \right) \quad (x \geq 2)$$

and prove **one** direction of this equivalence.

Solution. We claim that (1) is equivalent to

$$(2) \quad S(x) = 2\sqrt{x} \left(1 + O\left(\frac{1}{\log x}\right) \right) \quad (x \geq 2).$$

Assuming (1), we have, by partial summation

$$\begin{aligned} S(x) &= \frac{\pi(x) \log x}{\sqrt{x}} - \int_2^x (t^{-3/2} + (-1/2)t^{-3/2} \log t) \pi(t) dt \\ &= \sqrt{x} \left(1 + O\left(\frac{1}{\log x}\right) \right) + (1/2) \int_2^x t^{-1/2} dt + O\left(\int_2^x \frac{1}{\sqrt{t} \log t} dt\right) \\ &= \sqrt{x} \left(1 + O\left(\frac{1}{\log x}\right) \right) + \sqrt{x} + O\left(\frac{\sqrt{x}}{\log x}\right), \end{aligned}$$

where the last integral is estimated by

$$\int_2^x \frac{1}{\sqrt{t} \log t} dt \leq \frac{1}{\log \sqrt{x}} \int_{\sqrt{x}}^x \frac{1}{\sqrt{t}} dt + \frac{1}{\log 2} \int_2^{\sqrt{x}} \frac{1}{\sqrt{t}} dt \ll \frac{\sqrt{x}}{\log x}.$$

This proves (2).

The proof of the converse direction, (2) \Rightarrow (1), is analogous.

Problem 5

(i) State an integral representation for $\zeta(s)$ that is valid in the half-plane $\sigma > 0$. (No proofs are expected for this part, but if you don't recall the exact form of the identity, you may want to rederive it (which is not hard).)

(ii) Obtain a bound of the form $|\zeta(1/2 + it)| \leq c|t|$ for $|t| \geq 2$ with an explicit (i.e., numerical) constant c .

Solution. (i) The integral representation sought, valid in the half-plane $\sigma > 0$, is

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \{u\} u^{-s-1} du.$$

(ii) If $s = 1/2 + it$ with $|t| \geq 2$, then $|s| \leq 1/2 + |t| \leq 2|t|$, $|s-1| \geq |\sigma-1| = 1/2$ and $\int_1^\infty |\{u\} u^{-s-1}| du \leq \int_1^\infty u^{-3/2} du = 2$, so from the above representation we get

$$|\zeta(s)| \leq \left| \frac{s}{s-1} \right| + |s| \int_1^\infty |\{u\} u^{-s-1}| du \leq \frac{2|t|}{1/2} + 2|t| \cdot 2 = 8|t|.$$

Problem 6

Let $\omega = e^{2\pi i/3}$, and let $f(n) = \omega^n$.

(i) Determine the abscissa of convergence σ_c of the Dirichlet series $F(s) = \sum_{n=1}^\infty f(n)n^{-s}$.

(ii) Determine whether the series $\sum_p f(p)p^{-1}$ converges.

Solution. (i) Note that $f(n)$ is periodic with period 3, and $\sum_{n=1}^3 f(n) = \sum_{n=1}^3 \omega^n = 1 + e^{2\pi i/3} + e^{-2\pi i/3} = 0$, so the partial sums $S(x) = \sum_{n \leq x} f(n)$ satisfy $(*) |S(x)| \leq \sum_{n=1}^3 |f(n)| = 3$ for all $x \geq 1$. By partial summation, we have, for any complex number s and any positive integer N ,

$$(1) \quad \sum_{n=1}^N \frac{f(n)}{n^s} = \frac{S(N)}{N^s} + s \int_1^N S(x) x^{-s-1} dx.$$

Suppose now that $\sigma > 0$ and let $N \rightarrow \infty$ in (1). By $(*)$ the right-hand side converges to the (convergent) integral $\int_1^\infty S(x) x^{-s-1} dx$. Hence the left-hand side converges as well, i.e., the infinite series $F(s) = \sum_{n=1}^\infty f(n)n^{-s}$ converges. This shows that $\sigma_c \leq 0$. Since $|f(n)| = 1$ for all n , we have trivially $\sigma_c \geq 0$, so $\sigma_c = 0$.

(ii) We group the terms in this series according to the residue class of p modulo 3. Since all primes other than 3 are congruent to 1 or 2 modulo 3, we obtain for the partial sums of the given series

$$\sum_{p \leq x} \frac{f(p)}{p} = \frac{\omega^3}{3} + \sum_{a=1}^2 \omega^a \sum_{p \leq x, p \equiv a \pmod{3}} \frac{1}{p}.$$

By the quantitative form of Dirichlet's theorem, the latter is equal to

$$\begin{aligned} \frac{1}{3} + \sum_{a=1}^2 \frac{\omega^a}{\phi(3)} \log \log x + O(1) \\ = (\omega + \omega^2) \frac{1}{2} \log \log x + O(1) \\ = -\frac{1}{2} \log \log x + O(1). \end{aligned}$$

Hence the series $\sum_p f(p)p^{-1}$ diverges (to $-\infty$).

Problem 7

Let $f(n)$ denote the number of representations of n in the form $n = ab$, with **relatively prime** positive integers a, b .

(i) Determine the abscissa of convergence of the Dirichlet series $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$, and express this series in terms of the Riemann zeta function.

(ii) Obtain an estimate for $\sum_{n \leq x} f(n)$ with as good an error term as you can get. (Full credit for an estimate with error term $O(\sqrt{x} \log x)$; partial credit may be given for a weaker error estimate.) (You can leave constants appearing in this estimate unspecified.)

Solution. (i) Note that f is multiplicative, with $f(p^m) = 2$ for all $m \geq 1$ (since the only admissible factorizations $p^m = ab$ with $(a, b) = 1$ are $(a, b) = (p^m, 1)$ and $(a, b) = (1, p^m)$). Since f is multiplicative, $F(s)$ has an Euler product given by

$$\begin{aligned} F(s) &= \prod_p \left(1 + \sum_{m \geq 1} \frac{2}{p^{ms}} \right) \\ &= \prod_p \left(1 + \frac{2}{p^s - 1} \right) = \prod_p \frac{1 + p^{-s}}{1 - p^{-s}} = \prod_p \frac{1 - p^{-2s}}{(1 - p^{-s})^2} = \frac{\zeta(s)^2}{\zeta(2s)}. \end{aligned}$$

Since

$$\begin{aligned} \sum_p \sum_{m \geq 1} \left| \frac{f(p^m)}{p^{ms}} \right| &= \sum_p \sum_{m \geq 1} \frac{2}{p^{m\sigma}} \\ &= \sum_p \sum_{m \geq 1} \frac{2}{p^{m\sigma}} \ll \sum_p \frac{1}{p^\sigma} < \infty \quad (\sigma > 1), \end{aligned}$$

the Euler product converges absolutely in the half-plane $\sigma > 1$, and hence $F(s)$ converges absolutely in the same half-plane. Thus, the abscissa of

absolute convergence σ_a of $F(s)$ satisfies $\sigma_a \leq 1$. On the other hand, since f is nonnegative and $f(n) \geq 1$, we have $\sigma_c = \sigma_a \geq 1$. Hence $\sigma_c = \sigma_a = 1$.

(ii) Let $S(x) = \sum_{n \leq x} f(n)$. We will show that, with certain constants A and B ,

$$(1) \quad S(x) = Ax \log x + Bx + O(\sqrt{x} \log x) \quad (x \geq 2).$$

This is an estimate of the desired form, i.e., with *relative* error $O(1/\sqrt{x})$.

Since $f(p) = 2 = d(p)$ for primes p (where d is the divisor function), and a good estimate for the summatory function of $d(n)$ is available, namely Dirichlet's estimate

$$(2) \quad D(x) = \sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}) \quad (x \geq 1).$$

we apply the convolution method with $f_0(n) = d(n)$ as the approximating function. Accordingly, we define g by $f = f_0 * g$. Since f and f_0 are multiplicative, g is multiplicative as well, and a simple calculation shows that $g(p^m) = -1$ if $m = 2$ and $g(p^m) = 0$ otherwise. Thus,

$$(3) \quad g(n) = \begin{cases} \mu(m) & \text{if } n = m^2, \\ 0 & \text{otherwise.} \end{cases}$$

For $x \geq 1$ we have

$$\begin{aligned} S(x) &= \sum_{n \leq x} f(n) = \sum_{n \leq x} \sum_{d|n} g(d) f_0(n/d) \\ &= \sum_{d \leq x} g(d) \sum_{n \leq x/d} f_0(n) = \sum_{d \leq x} g(d) D(x/d). \end{aligned}$$

Substituting the estimate (2) for $D(x/d)$, we get

$$(4) \quad \begin{aligned} S(x) &= (x \log x) \sum_{d \leq x} \frac{g(d)}{d} - x \sum_{d \leq x} \frac{g(d) \log d}{d} \\ &\quad + (2\gamma - 1)x \sum_{d \leq x} \frac{g(d)}{d} + O\left(\sqrt{x} \sum_{d \leq x} \frac{|g(d)|}{\sqrt{d}}\right). \end{aligned}$$

The sums involving $g(d)$ can be evaluated as follows, using the formula (3):

$$(5) \quad \begin{aligned} \sum_{d \leq x} \frac{g(d)}{d} &= \sum_{m \leq \sqrt{x}} \frac{\mu(m)}{m^2} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} - \sum_{m > \sqrt{x}} \frac{\mu(m)}{m^2} \\ &= \frac{6}{\pi^2} + O\left(\frac{1}{\sqrt{x}}\right) \quad (x \geq 1), \end{aligned}$$

since $\sum_{n \geq y} n^{-2} = O(1/y)$ for $y \geq 1$; and

$$\begin{aligned}
 (6) \quad \sum_{d \leq x} \frac{g(d) \log d}{d} &= \sum_{m \leq \sqrt{x}} \frac{\mu(m) \log m^2}{m^2} \\
 &= 2 \sum_{m=1}^{\infty} \frac{\mu(m) \log m}{m^2} - \sum_{m > \sqrt{x}} \frac{\mu(m) \log m}{m^2} \\
 &= C + O\left(\frac{\log x}{\sqrt{x}}\right) \quad (x \geq 2),
 \end{aligned}$$

where C is a constant.

$$\begin{aligned}
 \sum_{n > y} \frac{\log n}{n^2} &\ll \sum_{y < n \leq y^2} \frac{\log y^2}{n^2} + \sum_{n > y^2} \frac{n^{1/2}}{n^2} \\
 &= O\left(\frac{\log y}{y}\right) + O\left(\frac{1}{(y^2)^{1/2}}\right) = O\left(\frac{\log y}{y}\right) \quad (y \geq 2).
 \end{aligned}$$

Substituting (5) and (6) into (3) we get the asserted estimate (1) with $A = 6/\pi^2$ and $B = -C + (2\gamma - 1)\pi^2/6$.

Remarks. If this problem looked vaguely familiar, this is no accident. The function f is, in fact, $2^{\omega(n)}$, and the estimation of the sum $\sum_{n \leq x} 2^{\omega(n)}$ was a homework problem.