# Analytic Number Theory
# Problem Set 1
# Due September 12, 2005

## General Remarks

**About these problems.** Some problems are harder than others, some are more interesting than others, but all are "doable", and the majority are at a level comparable to that of a Comprehensive Exam problem. Problems marked by an asterisk or labeled as bonus problems are of a less routine variety and usually beyond the exam level. In case of a particularly good/clever/original/elegant solution on such a problem I may give extra credit.

At the end of the assignment are suggestions of additional practice problems from Apostol's text (which is on reservere in the library), usually routine/quickie type problems intended as "warmups". Those problems are not to be turned in.

**Grading:** Most problems will be graded on a 5 point scale. You should do all problems, but I will select 4 or 5 problems from each assignment to be graded. (This is, unfortunately, a necessity because of the size of this class.) The graded problems will not be announced in advance, and will be the same for every student. They will consist of a mix of routine and less routine problems and are intended to be representative of the entire assignment.

**Write-up of solutions.** For all problems you need to provide complete, rigorous proofs; answers alone don't count. Try to write up your argument in a concise, but clear manner, without omitting any details. Write in complete grammatical sentences, but avoid being excessively verbose or rambling. Writing up mathematics is a skill that has to be learned, and classes like this one serve to practice and hone that skill.

The write-up of your solution will be considered in the grading; you will not get full credit if your solution is imprecise, confusing, or has gaps. What counts is not whether or not you "know" how to do a problem, but whether you can present the solution in a clear, convincing manner.

### Problem 1

Evaluate the function $f(n) = \sum_{d^2|n} \mu(d)$ (where the summation runs over all positive integers $d$ such that $d^2|n$), in the sense of expressing it in terms of familiar arithmetic functions. Be sure to give a complete proof.

### Problem 2

Determine an arithmetic function $f$ such that

$$\frac{1}{\phi(n)} = \sum_{d|n} \frac{1}{d} f\left(\frac{n}{d}\right) \quad (n \in \mathbb{N}).$$

### Problem 3

Let $f(n) = \#\{(n_1, n_2) \in \mathbb{N}^2 : [n_1, n_2] = n\}$, where $[n_1, n_2]$ is the least common multiple of $n_1$ and $n_2$. Show that $f$ is multiplicative and evaluate $f$ at prime powers.

### Problem 4

Let $f(n) = \phi(n)/n$, and let $\{n_k\}_{k=1}^{\infty}$ be the sequence of values $n$ at which $f$ attains a "record low"; i.e., $n_1 = 1$ and, for $k \geq 2$, $n_k$ is defined as the smallest integer $> n_{k-1}$ with $f(n_k) < f(n)$ for all $n < n_k$. (For example, since the first few values of the sequence $f(n)$ are $1, 1/2, 2/3, 1/2, 4/5, 1/3, \ldots$, we have $n_1 = 1$, $n_2 = 2$, and $n_3 = 6$, and the corresponding values of $f$ at these arguments are 1, 1/2 and 1/3.) Find (with proof) a general formula for $n_k$ and $f(n_k)$.

## Problem 5

A positive integer $n$ is called squarefull if it satisfies $(*)$ $p|n \Rightarrow p^2|n$. (Note that $n = 1$ is squarefull according to this definition, since 1 has no prime divisors and the above implication is therefore trivially true.) Show that $n$ is squarefull if and only if $n$ can be written in the form $n = a^2 b^3$ with $a, b \in \mathbb{N}$.
**Bonus question:** Find a similar characterization of "$k$-full" integers, i.e., integers $n \in \mathbb{N}$ that satisfy $(*)$ with 2 replaced by $k$ (where $k \geq 3$).

## Problem 6

Given an arithmetic function $f$ such that $\sum_{n=1}^{\infty} |f(n)| d(n) < \infty$, define its "transform" $\hat{f}$ by

$$\hat{f}(d) = \sum_{n=1}^{\infty} f(nd) \quad (d \in \mathbb{N}).$$

Find (with proof) the corresponding "inverse transform", i.e., a formula expressing $f(d)$ in terms of the values $\hat{f}(n)$.

## Problem 7*

Let $f$ be a multiplicative function satisfying $\lim_{p^m \to \infty} f(p^m) = 0$. Show that $\lim_{n \to \infty} f(n) = 0$.

## Problem 8*

Let $\mathcal{P} = \{p_1, \dots, p_k\}$ be a finite set of primes, let

$$\mathbb{N}_{\mathcal{P}} = \{n \in \mathbb{N} : p|n \Rightarrow p \in \mathcal{P}\}$$

i.e., $\mathbb{N}_{\mathcal{P}}$ is the set of positive integers all of whose prime factors belong to the set $\mathcal{P}$ (note that $1 \in \mathbb{N}_{\mathcal{P}}$), and let

$$N_{\mathcal{P}}(x) = \#\{n \in \mathbb{N}_{\mathcal{P}} : n \leq x\} \quad (x \geq 1).$$

In class it was shown that one has $N_{\mathcal{P}}(x) \leq c_1 (\log x)^k$ for a suitable constant $c_1$ (depending on the set $\mathcal{P}$, but not on $x$) and for all sufficiently large $x$, say $x \geq x_1$. This immediately implies that there must be infinitely many primes, since otherwise one could apply this result with $\mathcal{P}$ the set of **all** primes, and consequently $\mathbb{N}_\mathcal{P}$ the set of all positive integers, and would get a contradiction to the obvious fact that $\mathbb{N}_\mathcal{P}(x) = [x]$ when $\mathcal{P}$ consists of all primes. Show that a bound of the same type holds in the other direction, i.e., there exist constants $c_2 > 0$ and $x_2$, depending on $\mathcal{P}$, such that $N_\mathcal{P}(x) \geq c_2(\log x)^k$ holds for all $x \geq x_2$.

## Additional problems

One of the chief merits of Apostol's text (available on reserve in the library) is the large number of exercises that it contains. Below is a set of the problems from Chapter 2 of Apostol's text that I would recommend for additional practice. These problems are mostly routine drills, at an easy to medium level of difficulty (comparable to an exam problem), and they can all be done using only material we covered in class.

**Apostol, Chapter 2 (p. 46–51):** 1, 2, 3, 4, 5, 6 , 8*, 10, 11, 18, 19, 25, 27, 33.