Analytic Number Theory
Problem Set 2
Due September 26, 2005

Problem 1

Call a set of integers a *PC-set* if it has the property that any pair of distinct elements of the set is coprime. Given $x \geq 2$, let $N(x) = \max\{|A| : A \subset [2,x],\ A \text{ is a PC-set}\}$. In other words, $N(x)$ is the maximal number of integers with the PC property that one can fit in the interval $[2,x]$. Prove that $N(x)$ is equal to $\pi(x)$, the number of primes $\leq x$.

Problem 2

Let

$$I(x,\alpha) = \int_1^x \frac{\sin(\alpha t)}{t} dt,$$

where α is a fixed real and non-zero number. Use integration by parts to show that $I(x,\alpha)$ converges as $x \to \infty$, with limit $I(\alpha)$, say, and show that $I(x,\alpha) = I(\alpha) + O_\alpha(1/x)$.

Problem 3

Let $f(x)$ and $g(x)$ be positive, continuous functions on $[0,\infty)$, and set $F(x) = \int_0^x f(y)dy$, $G(x) = \int_0^x g(y)dy$.

(i) Show (by a counterexample) that the relation

$$f(x) = o(g(x)) \quad (x \to \infty) \tag{1}$$

does *not* imply

$$F(x) = o(G(x)) \quad (x \to \infty). \tag{2}$$

(ii) **Bonus question:** Find (with proof) an appropriate *general* condition on $g(x)$ under which the implication (1) $\Rightarrow$ (2) becomes true.

Remark: It is trivial to show that, if "o" is replaced by "O" in (1) and (2), then the implication holds. In other words, one can "pull out" a O-sign from an integral (provided the integrand is positive).

Problem 4

Obtain an estimate for the sum $\sum_{n \le x} (\log n)/n$ with error term $O((\log x)/x)$.

Problem 5

Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with nonnegative terms a_n.

(i) Show that there exists a real-valued function $\psi(n)$ with $\lim_{n\to\infty} \psi(n) = \infty$ such that the series $\sum_{n=1}^{\infty} a_n \psi(n)$ still converges.

(ii)* Show that the conclusion holds even without the assumption that the terms a_n be nonnegative (i.e., assuming only that the series $\sum_{n=1}^{\infty} a_n$ converges). (This requires a different, and more complicated argument.)

Problem 6*

Show that if $f(x)$ satisfies $f(x) = x^2 + O(x)$, and f is differentiable *with nondecreasing derivative* $f'(x)$ for sufficiently large x, then $f'(x) = 2x + O(\sqrt{x})$.

Remark. While O-estimates can be integrated provided the range of integration is contained in the range of validity of the estimate, in general such estimates cannot be differentiated. The above problem illustrates a situation where, under certain additional conditions (namely, the monoticity of the derivative), differentiation of a O-estimate is allowed.

Problem 7*

Let n be an integer ≥ 2 and p a positive real number. In class it was shown that (in the case $n = 2$, but the same argument works for general n)

$$\left(\sum_{i=1}^{n} a_i\right)^p \asymp_{n,p} \sum_{i=1}^{n} a_i^p \quad (a_1, a_2, \dots, a_n > 0).$$

By the definition of the notation $\asymp_{n,p}$, this means that there exist positive constants $c_1(n,p)$ and $c_2(n,p)$ such that

$$c_1(n,p) \sum_{i=1}^{n} a_i^p \leq \left(\sum_{i=1}^{n} a_i\right)^p \leq c_2(n,p) \sum_{i=1}^{n} a_i^p \quad (a_1, a_2, \dots, a_n > 0).$$

Determine the *best-possible* values for these constants.

Additional practice problems

Chapter 3 in Apostol's text provides plenty of additional practice material. In particular, the computation of the average orders of $d(n)$, $\sigma_\alpha(n)$ and $\phi(n)$ in Sections 3.5 - 3.7 illustrate the manipulation of double sums with divisibility conditions. Think of these theorems as exercises; there is no need to memorize the theorems (other than those that we covered in class), but you should be able to prove results of this kind on your own.

The general formula given in Section 3.10 will not be covered in class, and you need not memorize the formula. The main results of 3.11 will be covered later in class.

Among the exercises in Chapter 3 of Apostol, Problems 1–9 are mostly routine (though, in some cases, somewhat tedious) applications of one of the following methods: Euler's summation formula, summation by parts, the hyperbola method, and the convolution method. (Several problems, or variants thereof, are part of this assignment.) Problem 10 can be done with the convolution method (though I am not sure that this is what the author had in mind), and one then gets a much stronger result than the one claimed in the problem, namely that $\sum_{n\leq x} 1/\phi(n) = c_1 \log x + c_2 + O((\log x)/x)$. Problem 12 is an illustration of a trick used in class (more than once) to deal with summations involving a coprimality condition such as $(n,k) = 1$. The final group of exercises in this chapter (on the greatest integer functions) is not relevant for this course.