Analytic Number Theory
Problem Set 2
Due September 26, 2005

Problem 1

Call a set of integers a *PC-set* if it has the property that any pair of distinct elements of the set is coprime. Given $x \geq 2$, let $N(x) = \max\{|A| : A \subset [2, x],\ A \text{ is a PC-set}\}$. In other words, $N(x)$ is the maximal number of integers with the PC property that one can fit in the interval $[2, x]$. Prove that $N(x)$ is equal to $\pi(x)$, the number of primes $\leq x$.

Problem 2

Let

$$I(x, \alpha) = \int_1^x \frac{\sin(\alpha t)}{t} dt,$$

where α is a fixed real and non-zero number. Use integration by parts to show that $I(x, \alpha)$ converges as $x \to \infty$, with limit $I(\alpha)$, say, and show that $I(x, \alpha) = I(\alpha) + O_\alpha(1/x)$.

Problem 3

Let $f(x)$ and $g(x)$ be positive, continuous functions on $[0, \infty)$, and set $F(x) = \int_0^x f(y)dy$, $G(x) = \int_0^x g(y)dy$.

(i) Show (by a counterexample) that the relation

$$f(x) = o(g(x)) \quad (x \to \infty) \tag{1}$$

does *not* imply

$$F(x) = o(G(x)) \quad (x \to \infty). \tag{2}$$

(ii) **Bonus question:** Find (with proof) an appropriate *general* condition on $g(x)$ under which the implication (1) $\Rightarrow$ (2) becomes true.

Remark: It is trivial to show that, if "o" is replaced by "O" in (1) and (2), then the implication holds. In other words, one can "pull out" a O-sign from an integral (provided the integrand is positive).

Problem 4

Obtain an estimate for the sum $\sum_{n \leq x} (\log n)/n$ with error term $O((\log x)/x)$.

Problem 5

Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with nonnegative terms a_n.

(i) Show that there exists a real-valued function $\psi(n)$ with $\lim_{n\to\infty} \psi(n) = \infty$ such that the series $\sum_{n=1}^{\infty} a_n \psi(n)$ still converges.

(ii)* Show that the conclusion holds even without the assumption that the terms a_n be nonnegative (i.e., assuming only that the series $\sum_{n=1}^{\infty} a_n$ converges). (This requires a different, and more complicated argument.)

Problem 6*

Show that if $f(x)$ satisfies $f(x) = x^2 + O(x)$, and f is differentiable *with nondecreasing derivative* $f'(x)$ for sufficiently large x, then $f'(x) = 2x + O(\sqrt{x})$.

Remark. While O-estimates can be integrated provided the range of integration is contained in the range of validity of the estimate, in general such estimates cannot be differentiated. The above problem illustrates a situation where, under certain additional conditions (namely, the monoticity of the derivative), differentiation of a O-estimate is allowed.

Problem 7*

Let n be an integer ≥ 2 and p a positive real number. In class it was shown that (in the case $n = 2$, but the same argument works for general n)

$$\left(\sum_{i=1}^{n} a_i\right)^p \asymp_{n,p} \sum_{i=1}^{n} a_i^p \quad (a_1, a_2, \ldots, a_n > 0).$$

By the definition of the notation $\asymp_{n,p}$, this means that there exist positive constants $c_1(n,p)$ and $c_2(n,p)$ such that

$$c_1(n,p) \sum_{i=1}^{n} a_i^p \leq \left(\sum_{i=1}^{n} a_i\right)^p \leq c_2(n,p) \sum_{i=1}^{n} a_i^p \quad (a_1, a_2, \ldots, a_n > 0).$$

Determine the *best-possible* values for these constants.