Math 531, Fall 2005
Analytic Number Theory
Problem Set 2
Solutions

Problem 1

Call a set of integers a *PC-set* if it has the property that any pair of distinct elements of the set is coprime. Given $x \geq 2$, let $N(x) = \max\{|A| : A \subset [2,x],\ A \text{ is a PC-set}\}$. In other words, $N(x)$ is the maximal number of integers with the PC property that one can fit in the interval $[2,x]$. Prove that $N(x)$ is equal to $\pi(x)$, the number of primes $\leq x$.

Solution

Clearly the set P of primes is a PC-set, so $N(x) \geq |P \cap [2,x]| = \pi(x)$. Thus, it remains to show that $N(x) \leq \pi(x)$.

Let $A = \{a_1 < a_2 < \cdots < a_n\} \subset [2,x]$ be a PC-set. We need to show that $n \leq \pi(x)$. Let p_i denote the smallest prime factor of a_i. Since a_i is an integer in the interval $[2,x]$, p_i exists and is contained in the same interval. Moreover, for $i \neq j$ we have $p_i \neq p_j$ since otherwise a_i and a_j would have p_i as common prime factor, contradicting the PC property. Hence $p_1, p_2, \ldots, p_n$ represent n distinct primes $\leq x$, and so n cannot be larger than the total number of primes $\leq x$, i.e., we have $n \leq \pi(x)$, as claimed.

Problem 2

Let

$$I(x,\alpha) = \int_1^x \frac{\sin(\alpha t)}{t}\,dt,$$

where α is a fixed real and non-zero number. Use integration by parts to show that $I(x,\alpha)$ converges as $x \to \infty$, with limit $I(\alpha)$, say, and show that $I(x,\alpha) = I(\alpha) + O_\alpha(1/x)$.

Solution

Integration by parts gives, for any $x \geq 1$,

$$\begin{aligned} I(x,\alpha) &= \frac{-\cos(\alpha t)}{\alpha t}\Big|_1^x - \frac{1}{\alpha}\int_1^x \frac{\cos(\alpha t)}{t^2}dt \\ &= -\frac{\cos(\alpha x)}{\alpha x} + \frac{\cos(\alpha)}{\alpha} - \frac{1}{\alpha} I_1(x,\alpha), \end{aligned}$$

where $I_1(x,\alpha) = \int_1^x (\cos(\alpha t))t^{-2}dt$.

To estimate $I_1(x,\alpha)$ we use the following standard trick: instead of estimating the integral directly (which would only give a bound of order $O(1)$), we extend the range of integration to infinity and estimate instead the tail of the infinite integral. The infinite integral is absolutely convergent, since the integrand is bounded by $1/t^2$ and $\int_1^\infty (1/t^2)dt$ converges. If

$$I_1(\alpha) = \int_1^\infty \frac{\cos(\alpha t)}{t^2}dt$$

denotes this infinite integral, then

$$I_1(x,\alpha) = I_1(\alpha) - \int_x^\infty \frac{\cos(\alpha t)}{t^2}dt = I_1(\alpha) + O\left(\frac{1}{x}\right),$$

since

$$\left|\int_x^\infty \frac{(\cos(\alpha t)}{t^2}\right| \leq \int_x^\infty \frac{1}{t^2}dt = \frac{1}{x}.$$

Altogether, we have

$$I(x,\alpha) = \frac{\cos(\alpha)}{\alpha} - \frac{1}{\alpha}I_1(\alpha) + O_\alpha\left(\frac{1}{x}\right). \tag{1}$$

From this we deduce first that the limit $I(\alpha) = \lim_{x\to\infty} I(x,\alpha)$ exists and equals

$$I(\alpha) = \frac{\cos\alpha - I_1(\alpha)}{\alpha}.$$

Substituting this formula back into (1) then gives

$$I(x,\alpha) = I(\alpha) + O_\alpha\left(\frac{1}{x}\right),$$

which is an estimate of the desired form.

Problem 3

Let $f(x)$ and $g(x)$ be positive, continuous functions on $[0,\infty)$, and set $F(x) = \int_0^x f(y)dy$, $G(x) = \int_0^x g(y)dy$.

(i) Show (by a counterexample) that the relation

$$f(x) = o(g(x)) \quad (x \to \infty) \tag{1}$$

does *not* imply

$$F(x) = o(G(x)) \quad (x \to \infty). \tag{2}$$

(ii) **Bonus question:** Find (with proof) an appropriate *general* condition on $g(x)$ under which the implication (1) $\Rightarrow$ (2) becomes true.

Remark: It is trivial to show that, if "o" is replaced by "O" in (1) and (2), then the implication holds. In other words, one can "pull out" a O-sign from an integral (provided the integrand is positive).

Solution

(i) A counterexample is given by $f(y) = e^{-2y}$ and $g(y) = e^{-y}$. Since $f(y)/g(y) = e^{-y} \to 0$ as $y \to \infty$, we have $f(y) = o(g(y))$. On the other hand, $F(x) = (1 - e^{-2x})/2$ and $G(x) = 1 - e^{-x}$ converge to $1/2$ and 1, respectively, as $x \to \infty$, so $F(x) = o(G(x))$ certainly does not hold.

(ii) The general condition sought is

$$\int_0^\infty g(y)dy = \infty. \tag{3}$$

We will show that under this condition, (1) implies (2).

Let $\epsilon > 0$ be given. Since $f(x) = o(g(x))$ there exists a $y_0(\epsilon) > 0$ such that

$$f(y) \le \frac{\epsilon}{2} g(y) \quad (y \ge y_0(\epsilon)). \tag{4}$$

(Note that by hypothesis the functions f and g are positive-valued, so there is no need to use absolute value signs here.) Since the function f is continuous on $[0, \infty)$, it is bounded on the interval $[0, y_0(\epsilon)]$ and attains its maximum there. Let

$$M(\epsilon) = \max_{0 \le y \le y_0(\epsilon)} f(y).$$

Then, for $x \ge y_0(\epsilon)$,

$$\begin{aligned} F(x) &= \int_0^{y_0(\epsilon)} f(y)dy + \int_{y_0(\epsilon)}^x f(y)dy \\ &\le M(\epsilon) y_0(\epsilon) + \int_{y_0(\epsilon)}^x \frac{\epsilon}{2} g(y)dy \\ &\le M(\epsilon) y_0(\epsilon) + \frac{\epsilon}{2} G(x). \end{aligned}$$

Now, our additional assumption (3) means that $\lim_{x\to\infty} G(x) = \infty$. Thus, there exists an $x_0(\epsilon)$ such that

$$M(\epsilon)y_0(\epsilon) \le \frac{\epsilon}{2}G(x) \quad (x \ge x_0(\epsilon)).$$

For $x \ge \max(x_0(\epsilon), y_0(\epsilon))$ it therefore follows that $F(x) \le \epsilon G(x)$. Since $\epsilon > 0$ was arbitrary, this shows that $F(x) = o(G(x))$.

[An alternative, and shorter, proof can be given by applying l'Hopital's Rule. However, the above $\epsilon - x_0$ argument is more instructive, and more widely applicable.]

Problem 4

Obtain an estimate for the sum $\sum_{n\le x}(\log n)/n$ with error term $O((\log x)/x)$.

Solution

We will show that

$$\sum_{n\le x}\frac{\log n}{n} = \frac{1}{2}(\log x)^2 + A + O\left(\frac{\log x}{x}\right) \quad (x \ge 2), \tag{1}$$

where A is a constant. By the Euler-Maclaurin sum formula (in the special case for sums over all positive integers $n \le x$),

$$\sum_{n\le x}\frac{\log n}{n} = \frac{\log 1}{1} - \{x\}\frac{\log x}{x} + \int_1^x \frac{\log t}{t}dt + \int_1^x \{t\}\frac{1-\log t}{t^2}dt.$$

The first two terms on the right are of order $O((\log x)/x)$ for $x \ge 2$ (say); the third term can be evaluated exactly as

$$\int_1^x \frac{\log t}{t}dt = \int_0^{\log x} u du = \frac{1}{2}(\log x)^2;$$

and the last term represents a convergent integral when the integration is extended to infinity and thus can be written as

$$A - \int_x^\infty \{t\}\frac{1-\log t}{t^2}dt. \tag{2}$$

with $A = \int_1^\infty \{t\}\frac{1-\log t}{t^2}dt$. To prove (1), it remains to show that the integral in (2) is of order $O((\log x)/x)$ for $x \ge 2$. This can be seen as follows:

$$\begin{aligned}\left|\int_x^\infty \{t\}\frac{1-\log t}{t^2}dt\right| &\le (1+\log x^2)\int_x^{x^2}\frac{1}{t^2}dt + \int_{x^2}^\infty \frac{1+\log t}{t^2}dt \\ &\ll \frac{\log x^2}{x} + \int_{x^2}^\infty \frac{t^{1/2}}{t^2}dt = \frac{\log x}{x} + \frac{2}{3x} \ll \frac{\log x}{x}.\end{aligned}$$

Problem 5

Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with nonnegative terms a_n.

(i) Show that there exists a real-valued function $\psi(n)$ with $\lim_{n\to\infty} \psi(n) = \infty$ such that the series $\sum_{n=1}^{\infty} a_n\psi(n)$ still converges.

(ii)* Show that the conclusion holds even without the assumption that the terms a_n be nonnegative (i.e., assuming only that the series $\sum_{n=1}^{\infty} a_n$ converges). (This requires a different, and more complicated argument.)

Solution

(i) Since $\sum_{n=1}^{\infty} a_n$ is a convergent series, the tails of this series, $R_N = \sum_{n=N+1}^{\infty} a_n$, converge to 0 as $N \to \infty$. Thus, there exists a sequence of integers $N_1 < N_2 < N_3 < \cdots$ such that $R_{N_k} < 3^{-k}$ for each $k \geq 1$. Set $N_0 = 0$ and for $k = 0, 1, 2, \ldots$ let

$$\psi(n) = 2^k \quad (N_k < n \leq N_{k+1}). \tag{1}$$

This function is well-defined for all positive integers n and tends to infinity as $n \to \infty$. We now show that the series $\sum_{n=1}^{\infty} a_n\psi(n)$ converges. To do so, we will make use of the assumption that the terms a_n are nonnegative (as is the function $\psi(n)$ defined above). We have

$$\sum_{n=1}^{\infty} a_n\psi(n) = \sum_{k=0}^{\infty} 2^k \sum_{N_k < n \leq N_{k+1}} a_n \leq \sum_{k=0}^{\infty} R_{N_k} 2^k \leq R_{N_0} + \sum_{k=1}^{\infty} (2/3)^k < \infty, \tag{2}$$

since $R_{N_0} = \sum_{n=1}^{\infty} a_n$ is finite. Thus the series $\sum_{n=1}^{\infty} a_n\psi(n)$ converges.

(ii) Without the nonnegativity hypothesis, the result remains true. However, to prove the convergence of $\sum_{n=1}^{\infty} a_n\psi(n)$, we can no longer use the argument in (2), which depended on the fact that the series involved had only nonnegative terms. The idea is the same, but we have to argue more carefully, using Cauchy's criterion.

We define R_N by $R_N = \sup_{M>N} \left| \sum_{n=N+1}^{M} a_n \right|$. The convergence of $\sum_{n=1}^{\infty} a_n$ implies, by Cauchy's criterion, that $R_N \to 0$ as $N \to \infty$. Thus, as before, we can find a sequence $N_1 < N_2 < \cdots$ such that $R_{N_k} < 3^{-k}$ for all $k \geq 1$, and we again define $\psi(n)$ by (1). We will show that, with this function $\psi(n)$, $\sum_{n=1}^{\infty} a_n\psi(n)$ converges. To prove convergence of this series, we use Caucy's criterion, i.e., we show that for every $\epsilon > 0$ there exists $N_0(\epsilon)$ such that

$$\left| \sum_{n=N+1}^{M} a_n\psi(n) \right| < \epsilon \quad (M > N \geq N_0(\epsilon)). \tag{3}$$

Given $N < M$, let $k \le l$ be such that $N_k \le N < N_{k+1}$ and $N_l \le M < N_{l+1}$. Then,

$$\sum_{n=N+1}^{M} a_n \psi(n) = 2^k S' + \sum_{i=k+1}^{l-1} 2^i S_i + 2^l S'', \tag{4}$$

where

$$S_i = \sum_{n=N_i+1}^{N_{i+1}} a_n, \quad S' = \sum_{n=N+1}^{N_{k+1}} a_n, \quad S'' = \sum_{n=N_l+1}^{M} a_n.$$

The definition of R_N and N_i immediately give $|S_i| \le R_{N_i} \le 3^{-i}$ for each i and $|S''| \le R_{N_l} \le 3^{-l}$. Moreover, we have

$$|S'| \le |S_k| + \left| \sum_{n=N_k}^{N} a_n \right| \le 2R_k \le 2 \cdot 3^{-k}.$$

Thus, the sum on the left of (4) is

$$\le 2 \sum_{i \ge k} 2^i 3^{-i} \le 6 \cdot (2/3)^k.$$

This bound becomes $\le \epsilon$ if $k \ge k_0(\epsilon)$ with a sufficiently large $k_0(\epsilon)$. Hence, (3) holds if we choose $N_0(\epsilon) = N_{k_0(\epsilon)}$ (since this implies that for any $N \ge N_0(\epsilon)$ the number k defined by $N_k \le N < N_{k+1}$ is $\ge k_0(\epsilon)$).

Problem 6*

Show that if $f(x)$ satisfies $f(x) = x^2 + O(x)$, and f is differentiable *with non-decreasing derivative* $f'(x)$ for sufficiently large x, then $f'(x) = 2x + O(\sqrt{x})$.

Remark. While O-estimates can be integrated provided the range of integration is contained in the range of validity of the estimate, in general such estimates cannot be differentiated. The above problem illustrates a situation where, under certain additional conditions (namely, the monoticity of the derivative), differentiation of a O-estimate is allowed.

Solution

By the assumptions, there exists $x_0 > 0$ such that $f(x)$ is differentiable with nondecreasing derivative for $x \ge x_0$ and that $f(x) = x^2 + O(x)$ for $x \ge x_0$. Without loss of generality we may assume $x_0 \ge 2$. For $x \ge x_0$ we then have

$$f(x) = \int_{x_0}^{x} f'(t)dt + f(x_0).$$

Now, let $x \geq x_0$ and for $0 < y \leq x$ consider the difference $f(x+y) - f(x)$. One the one hand, by the O-estimate for f we have

(1) $f(x+y) - f(x) = (x+y)^2 - x^2 + O(x+y) + O(x) = 2xy + O(y^2) + O(x).$

On the other hand, the monotonicity of f' gives

$$f(x+y) - f(x) = \int_x^{x+y} f'(t)dt \geq yf'(x).$$

It follows that, for any $x \geq x_0$ and $0 < y \leq x$,

$$f'(x) \leq \frac{1}{y}\big(2xy + O(y^2) + O(x)\big) = 2x + O(y) + O(x/y) \quad (x \geq x_0, 0 < y \leq x).$$

Choosing $y = \sqrt{x}$ (which satisfies $y \leq x$ by our hypothesis $x \geq x_0 \geq 2$) gives

(1) $$f'(x) \leq 2x + O(\sqrt{x}) \quad (x \geq x_0).$$

Similarly, by considering the difference $f(x) - f(x-y)$ for $0 < y \leq x/2$ and $x \geq 2x_0 (\geq 4)$ we obtain

$$f'(x) \geq \frac{1}{y}(f(x) - f(x-y)) = 2x + O(y) + O(x/y) \quad (x \geq 2x_0, 0 < y \leq x/2).$$

As before, we choose $y = \sqrt{x}$. Our assumptions $x \geq 2x_0$ and $x_0 \geq 2$ ensure that this choice satisfies $y \leq x/2$ and $x - y \geq x/2 \geq x_0$. Thus, the given estimate for f is applicable to both $f(x-y)$ and $f(x)$, and we get

(2) $$f'(x) \geq 2x + O(\sqrt{x}) \quad (x \geq 2x_0).$$

(1) and (2) prove the asserted estimate.

Problem 7*

Let n be an integer ≥ 2 and p a positive real number. In class it was shown that (in the case $n = 2$, but the same argument works for general n)

$$\left(\sum_{i=1}^n a_i\right)^p \asymp_{n,p} \sum_{i=1}^n a_i^p \quad (a_1, a_2, \dots, a_n > 0).$$

By the definition of the notation $\asymp_{n,p}$, this means that there exist positive constants $c_1(n,p)$ and $c_2(n,p)$ such that

$$c_1(n,p) \sum_{i=1}^n a_i^p \leq \left(\sum_{i=1}^n a_i\right)^p \leq c_2(n,p) \sum_{i=1}^n a_i^p \quad (a_1, a_2, \dots, a_n > 0).$$

Determine the *best-possible* values for these constants.

Solution

We will show that

$$c_1(n,p) = \begin{cases} n^{p-1} & \text{if } 0 < p \le 1, \\ 1 & \text{if } p > 1, \end{cases}$$

and

$$c_2(n,p) = \begin{cases} 1 & \text{if } 0 < p \le 1, \\ n^{p-1} & \text{if } p > 1. \end{cases}$$

Given numbers $a_1, \dots, a_n > 0$, we set

$$S_p = \sum_{i=1}^n a_i^p, \quad S = S_1.$$

so that $c_1 = c_1(n,p)$ and $c_2 = c_2(n,p)$ are the best-possible constants in the inequalities

$$c_1 S_p \le S^p \le c_2 S_p.$$

For $p = 1$, we have $S_p = S$ and $c_1 = c_2 = 1$, so both inequalities hold trivially. Thus we may assume that $p \neq 1$. We distinguish between the two cases $0 < p < 1$ and $p > 1$. With the above definitions of c_1 and c_2, the inequalities to be shown break down into the following four cases:

$$S^p \le S_p \quad \text{if } 0 < p < 1, \tag{1}$$

$$S^p \le n^{p-1} S_p \quad \text{if } p > 1, \tag{2}$$

$$S_p \le n^{1-p} S^p \quad \text{if } 0 < p < 1, \tag{3}$$

$$S_p \le S^p \quad \text{if } p > 1. \tag{4}$$

Proof of (1). Set $x_i = a_i/S$, so that $0 \le x_i \le 1$ for each i and $\sum_{i=1}^n x_i = 1$. Since $0 < p < 1$, it follows that $x_i \le x_i^p$ and so

$$1 = \sum_{i=1}^n x_i \le \sum_{i=1}^n x_i^p = \sum_{i=1}^n (a_i/S)^p = S^{-p} S_p,$$

which implies (1).

Proof of (2). If $p > 1$, then by Hölder's inequality with summands $a_i = a_i \cdot 1$ and exponents p and $q = (1 - 1/p)^{-1}$ we have

$$S \le \left(\sum_{i=1}^n a_i^p \right)^{1/p} \left(\sum_{i=1}^n 1^q \right)^{1/q} = \left(\sum_{i=1}^n a_i^p \right)^{1/p} n^{1/q} = n^{1/q} S_p^{1/p}.$$

Taking the pth power on both sides and noting that $p/q = p(1 - 1/p) = p - 1$ then gives (2).

Proof of (3). We again apply Hölder's inequality, this time with summands $a_i^p = a_i^p \cdot 1$ and exponents $p' = 1/p$ and $q' = (1 - 1/p')^{-1} = (1-p)^{-1}$. Note that $p' > 1$ since $0 < p < 1$. We get

$$S_p \leq \left(\sum_{i=1}^{n} (a_i^p)^{p'} \right)^{1/p'} \left(\sum_{i=1}^{n} 1^{q'} \right)^{1/q'} = n^{1-p} S^p,$$

proving (3).

Proof of (4). This is similar to the proof of (1). Setting $x_i = a_i^p / S_p$, we have $0 \leq x_i \leq 1$ for each i, $\sum_{i=1}^{n} x_i = 1$, and $x_i \leq x_i^{1/p}$ since $p > 1$. Thus,

$$1 = \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} x_i^{1/p} = \sum_{i=1}^{n} a_i / S_p^{1/p} = S S_p^{-1/p},$$

and therefore $S_p \leq S^p$, which is (4).

Proof of the optimality of the constants. Choosing $a_i = 1$ for all i, we have $S_p = n$ and $S^p = n^p$. In this case the inequalities in (2) and (3) become equalities, so the constants in (2) and (3) are best-possible.

If we take $a_1 = 1$ and $a_i = 0$ for $i \neq 1$, then $S = S_p = 1$, and thus equality holds in (1) and (4). This choice has the defect that it does not satisfy the hypothesis $a_i > 0$. However, we can remedy this by perturbing these a_i's slightly, taking $(a_1, \ldots, a_n) = (1, \epsilon, \ldots, \epsilon)$ for some small $\epsilon > 0$. If $S(\epsilon)$ and $S_p(\epsilon)$ denote the sums corresponding to these tuples, then both $S(\epsilon)$ and $S_p(\epsilon)$ are continuous functions of ϵ, and so $\lim_{\epsilon \to 0+} S(\epsilon)/S_p(\epsilon) = S(0)/S_p(0) = 1$. Thus, the constant 1 in (1) and (4) is best-possible, even under the restriction that $a_i > 0$ for all i.

Remark: The fact that the proofs of (3) and (4) were very similar to those of (2) and (1) suggests that there ought to be a common approach to both sets of inequalities. Indeed, one can derive (3) and (4) from (2) and (1), respectively, by applying the latter inequalities with $p' = 1/p$ and $a_i' = a_i^p$, which has the effect of interchanging the roles of S and S_p.