Analytic Number Theory
Problem Set 3
Due October 10, 2005

Problem 1

Obtain an asymptotic estimate with error term $O(x^{1/3})$ for the number of squarefull integers $\le x$, i.e., for the quantity

$$S(x) = \#\{n \le x : p|n \Rightarrow p^2|n\}.$$

Problem 2

Given an arithmetic function $a(n)$, $n = 1, 2, \ldots$, and a real number $\alpha > -1$ define a mean value $M_\alpha(a)$ by

$$M_\alpha(a) = \lim_{x\to\infty} \frac{1+\alpha}{x^{1+\alpha}} \sum_{n\le x} n^\alpha a(n),$$

provided the limit exist. (In particular, $M_0(a) = M(a)$ is the usual asymptotic mean value of a.) Prove, using a rigorous $\epsilon - x_0$ argument, that the mean value $M_\alpha(a)$ exists if and only if the ordinary mean value $M(a) = M_0(a)$ exists. (As a consequence, if one of the mean values $M_\alpha(a)$, $\alpha > -1$, exists, then all of these mean values exist.)

Problem 3

Let f be an arithmetic function having a non-zero mean value $M(f) = A$, and let α be a fixed real number. Obtain an asymptotic formula for the sums $\sum_{n \le x} f(n) n^{i\alpha}$.

Problem 4

Obtain an estimate, similar to Dirichlet's estimate for $\sum_{n \le x} d(n)$, for the sum $\sum_{n \le x} 2^{\omega(n)}$. (You can leave constants appearing in this estimate unspecified.)

Problem 5

Using the Dirichlet hyperbola method (or some other method), obtain an estimate for the sum $\sum_{n\le x} d(n)/n$ with an error term $O((\log x)/\sqrt{x})$. (Note the error term. Exercise 2 in Section 3 of Apostol asks for an estimate of the same sum, but only with error $O(1)$.)

Problem 6

Let $q_1 = 1, q_2 = 2, q_3 = 3, q_4 = 5\dots$ denote the sequence of squarefree numbers.

(i) Obtain an asymptotic estimate with error term $O(\sqrt{n})$ for q_n.

(ii) Show that there are arbitrarily large gaps in the sequence $\{q_n\}$, i.e., $\limsup_{n\to\infty}(q_{n+1} - q_n) = \infty$. (Hint: Chinese Remainder Theorem.)

(iii) (Bonus problem) Prove the stronger bound

$$\limsup_{n\to\infty} \frac{q_{n+1} - q_n}{\log n / \log\log n} \ge \frac{1}{2}.$$

(iv) (Superbonus problem—deadline 10/31/05) Prove that (iii) holds with $1/2$ replaced by the constant $\pi^2/12$, i.e., that the limsup above is at least $\pi^2/12$.

(According to the famous number theorist Paul Erdös, this result can be proved by "a simple argument using the Chinese Remainder Theorem and the Prime Number Theorem", so all you need to do to earn the superbonus points is to supply this "simple argument" ....)

Problem 7

Show that, $\phi(n) \geq n/4$ holds for at least 1/3 of all positive integers n (in the sense that if A is the set of such n, then $\liminf_{x\to\infty}(1/x)\#\{n \leq x : n \in A\} \geq 1/3$). (Hint: use the fact that (1) $\sum_{n\leq x} \phi(n) \sim (3/\pi^2)x^2$ (which was proved in class) or (2) $\sum_{n\leq x} \phi(n)/n \sim (6/\pi^2)x$ (an easy consequence of Wintner's theorem, or of (1), by partial summation).)

Problem 8

Euler's proof of the infinitude of primes shows that $(*)$ $\sum_{p\le x} 1/p \ge \log\log x - C$, for some constant C and all sufficiently large x. This is a remarkably good lower bound for the sum of reciprocals of primes (it is off by only a term $O(1)$), so it is of some interest to see what this bound implies for $\pi(x)$. The answer is, surprisingly little, as the following problems show.

(i) Deduce from $(*)$, *without using any other information about the primes*, that there exists $\delta > 0$ such that $\pi(x) > \delta \log x$ for all sufficiently large x. In other words, show that if A is any sequence of positive integers satisfying

$$\sum_{a\le x, a\in A} \frac{1}{a} \ge \log\log x - C \tag{1}$$

for some constant C and all sufficiently large x, then there exists a constant $\delta > 0$ such that the counting function $A(x) = \#\{a \in A, a \le x\}$ satisfies

$$A(x) \ge \delta \log x \tag{2}$$

for all sufficiently large x.

(ii) (Superbonus problem—deadline 10/31/05) Show that this result is nearly best possible, in the sense that it becomes false if the function $\log x$ on the right-hand side of (2) is replaced by a power $(\log x)^\alpha$ with an exponent α greater than 1. In other words, given $\epsilon > 0$, construct a sequence A of positive integers, satisfying (1) above, but for which the counting function $A(x)$ $A(x) = \#\{a \in A, a \le x\}$ satisfies

$$\liminf_{x\to\infty} A(x) \log x)^{-1-\epsilon} \le 1. \tag{3}$$