Analytic Number Theory
Problem Set 4
Due October 17, 2005
(Note the early deadline)

Problem 1

Let $L(n) = [1, 2, \ldots, n]$, where $[\ldots]$ denotes the least common multiple. Show that the limit $\lim_{n\to\infty} L(n)^{1/n}$ exists if and only if the PNT holds.

Problem 2

Show that the estimate (a stronger version of one of Mertens' estimates)

$$\sum_{n\le x} \frac{\Lambda(n)}{n} = \log x + C + o(1), \qquad (1)$$

where C is a constant, implies the PNT.

Bonus problem. Using only methods and results discussed so far in class, show that the converse also holds, i.e., the PNT implies (1).

Problem 3

Let a_n be a nonincreasing sequence of positive numbers. Show that $\sum_p a_p$ converges if and only if $\sum_{n=2}^{\infty} a_n/\log n$ converges.

Problem 4

For positive integers k define the generalized von Mangoldt functions Λ_k by the identity $\sum_{d|n} \Lambda_k(d) = (\log n)^k$ (which for $k = 1$ reduces to the familiar identity for the ordinary von Mangoldt function $\Lambda(n)$). Show that $\Lambda_k(n) = 0$ if n has more than k distinct prime factors.

Problem 5

Call positive integer n round if it has no prime factors greater than $\sqrt{n}$. Let $R(x)$ denote the number of round integers $\leq x$. Estimate $R(x)$ to within an error $O(x/\log x)$. (Hint: Estimate first the slightly different counting function

$$R_0(x) = \#\{n \leq x : p|n \Rightarrow p \leq \sqrt{x}\},$$

and then show that the difference between $R(x)$ and $R_0(x)$ is of order $O(x/\log x)$ and thus negligible.)