Analytic Number Theory
Problem Set 4
Solutions

Problem 1

Let $L(n) = [1, 2, \ldots, n]$, where $[\ldots]$ denotes the least common multiple. Show that the limit $\lim_{n\to\infty} L(n)^{1/n}$ exists if and only if the PNT holds.

Solution

The desired equivalence is an immediate consequence of the identity $(*)$ $L(n) = e^{\psi(n)}$, which we will prove below, and the fact (proved in class) that the existence of the limit $\lim_{x\to\infty} \psi(x)/x$ is equivalent to the PNT.

To prove $(*)$, first note that

$$\exp(\psi(n)) = \exp\left(\sum_{p^m \le n} \log p\right) = \prod_{p\le n} \prod_{1\le m\le \log n/\log p} p = \prod_{p\le n} p^{\alpha_n(p)},$$

where $\alpha_n(p) = [\log n/\log p]$. On the other hand, for a given prime $p \le n$, the largest power of p dividing $L(n) = [1, 2, \ldots, n]$ is obviously p^m with m the largest exponent such that $p^m \le n$, i.e., $m = \alpha_n(p)$. Hence, $L(n)$ and $e^{\psi(n)}$ have the same prime factorization and therefore must be equal. This proves $(*)$.

Problem 2

Show that the estimate (a stronger version of one of Mertens' estimates)

$$\sum_{n\le x} \frac{\Lambda(n)}{n} = \log x + C + o(1), \tag{1}$$

where C is a constant, implies the PNT.

Bonus problem. Using only methods and results discussed so far in class, show that the converse also holds, i.e., the PNT implies (1).

Solution

(i) Assume that (1) holds, and set

$$\psi_1(x) = \sum_{n \le x} \frac{\Lambda(n)}{n},$$

so that by (1)

$$\psi_1(x) = \log x + C + r(x), \tag{2}$$

where C is a constant in (1) and $r(x)$ satisfies $r(x) = o(1)$. We will prove the PNT in the form $\psi(x) = x + o(x)$.

By partial summation we obtain, for any $x \ge 1$,

$$\begin{aligned}
\psi(x) = \sum_{n \le x} \frac{\Lambda(n)}{n} \cdot n &= \psi_1(x)x - \int_1^x \psi_1(t)dt \\
&= x \log x + Cx + r(x)x - \int_1^x (\log t + C + r(t))dt. \\
&= x \log x + Cx + r(x)x - (x \log x - x + 1) - C(x-1) - \int_1^x r(t)dt \\
&= x + r(x)x + \int_1^x r(t)dt + O(1).
\end{aligned}$$

The term $r(x)x$ is by hypothesis of order $o(x)$, as is (trivially) the term $O(1)$, so it remains to show that the same holds for the integral $\int_1^x r(t)dt$.

Let $\epsilon > 0$ given. By our assumption $r(x) = o(1)$ there exists $x_0 = x_0(\epsilon) \ge 1$ such that $|r(x)| \le \epsilon$ for $x \ge x_0$. Therefore, for $x \ge x_0$, we have

$$\left| \int_1^x r(t)dt \right| \le \int_1^{x_0} |r(t)|dt + \int_{x_0}^x \epsilon dt = O_\epsilon(1) + \epsilon x.$$

Since $\epsilon > 0$ was arbitrary, we conclude that

$$\limsup_{x \to \infty} \frac{1}{x} \left| \int_1^x r(t)dt \right| = 0,$$

which is what we wanted to prove.

(ii) In order to prove the converse direction, i.e., to deduce the relation (1) from the PNT, we have to proceed differently, as partial summation would only give an estimate that is much weaker than (1). Indeed, expressing $\psi_1(x)$ in terms of $\psi(x)$ by partial summation, we get

$$\psi_1(x) = \frac{\psi(x)}{x} + \int_1^x \frac{\psi(t)}{t^2} dt,$$

and using the PNT in the form $\psi(t) = t + o(t)$ on the right-hand side, gives an estimate for $\psi_1(x)$ involving an error $o(\log x)$, but not $o(1)$ as needed for (1).

The argument we will use is a variation of the proof of Mertens' estimate for $\sum_{n\le x} \Lambda(n)/n$, the difference being that the convolution sum arising in the process needs to be estimated more carefully, using the Dirichlet hyperbola method and exploiting the more precise information about $\psi(x)$ that is provided by the PNT.

Our starting point is an evaluation of the sum

$$S(x) = \sum_{n\le x} \log n$$

in two ways: On the one hand, a direct estimation of this sum gives

$$S(x) = x\log x - x + O(\log x). \tag{3}$$

On the other hand, writing $\log = \Lambda * 1$, we get

$$S(x) = \sum_{d\le x} \Lambda(d) \sum_{m\le x/d} 1. \tag{4}$$

Estimating the latter sum as

$$S(x) = \sum_{d\le x} \Lambda(d) \left(\frac{x}{d} + O(1)\right) = x \sum_{d\le x} \frac{\Lambda(d)}{d} + O(\psi(x)),$$

and using Chebyshev's bound $\psi(x) = O(x)$ would then give Mertens' estimate $\sum_{d\le x} \Lambda(d)/d = \log x + O(1)$. In order to obtain the stronger estimate (1), we need to replace the term $O(1)$ here by a term of the form $C + o(1)$, with a suitable constant C.

To this end, we split the double sum in (4) according to the Dirichlet hyperbola method. Given any $y \ge 2$ and $x \ge y$, we have

$$S(x) = \sum_{d\le x} \Lambda(d) \sum_{m\le x/d} 1 = S_1(x,y) + S_2(x,y) - S_3(x,y), \tag{5}$$

where

$$S_1(x,y) = \sum_{d \le x/y} \Lambda(d) \sum_{m \le x/d} 1,$$
$$S_2(x,y) = \sum_{m \le y} 1 \sum_{d \le x/m} \Lambda(d),$$
$$S_3(x,y) = \sum_{d \le x/y} \Lambda(d) \sum_{m \le y} 1.$$

We now estimate each of these three sums.

First, setting

$$\psi_1(x) = \log x + R(x),$$

we have

$$\begin{aligned} S_1(x,y) &= \sum_{d \le x/y} \Lambda(d) \left(\frac{x}{d} + O(1) \right) \\ &= x\psi_1(x/y) + O(\psi(x/y)) \\ &= x \log(x/y) + xR(x/y) + O(x/y), \end{aligned} \tag{6}$$

where, here and in the remainder of the proof, the O-constants are absolute (and so, in particular, independent of y). Next, defining $\delta(x)$ by

$$\psi(x) = x(1 + \delta(x)),$$

and setting

$$\delta^*(x) = \sup_{t \ge x} |\delta(t)|,$$

we have

$$\begin{aligned} S_2(x,y) &= \sum_{m \le y} \psi(x/m) = \sum_{m \le y} \frac{x}{m} (1 + \delta(x/m)) \\ &= x (\log y + \gamma + O(1/y)) + O(\delta^*(x/y) x \log y) \end{aligned} \tag{7}$$

and

$$\begin{aligned} S_3(x,y) &= (y + O(1))\psi(x/y) = (1 + O(1/y))\, x(1 + \delta(x/y)) \\ &= x + O(x/y) + O(\delta^*(x/y)x). \end{aligned} \tag{8}$$

Substituting (6)–(8) into (5), we get

$$S(x) = x \left(\log x + \gamma - 1 + R(x/y) + O(\delta^*(x/y) \log y) \right).$$

Comparing this estimate with (3), we obtain, after dividing by x,

$$R(x/y) + \gamma = O(1/y) + O(\delta^*(x/y)\log y). \tag{9}$$

Now let $x \to \infty$ in (9). Under the assumption of the PNT, we have $\psi(x) = x + o(x)$ as $x \to \infty$, so the functions $\delta(x)$ and $\delta^*(x)$ tend to 0 as $x \to \infty$. Thus, (9) implies

$$\limsup_{x\to\infty} |R(x/y) + \gamma| \le c\left(\frac{1}{y} + \limsup_{x\to\infty} \delta^*(x/y)\log y\right) = \frac{c}{y},$$

where c is an absolute constant. Replacing x by yx we get $\limsup_{x\to\infty} |R(x) + \gamma| \le c/y$. Since y can be chosen arbitrarily large, the limsup must be 0, and we conclude that $\lim_{x\to\infty} R(x) = -\gamma$. Since $R(x) = \psi_1(x) - \log x$, this proves (1) with constant $C = -\gamma$.

Problem 3

Let a_n be a nonincreasing sequence of positive numbers. Show that $\sum_p a_p$ converges if and only if $\sum_{n=2}^\infty a_n/\log n$ converges.

Solution

For any positive integer k, let

$$\begin{aligned} S_k &= \sum_{2^k < n \le 2^{k+1}} \frac{a_n}{\log n}, \\ T_k &= \sum_{2^k < p \le 2^{k+1}} a_p. \end{aligned}$$

Since all series involve only nonnegative terms, the convergence of $\sum_p a_p$ is equivalent to the convergence of $\sum_{k=1}^\infty T_k$, and that of $\sum_{n=2}^\infty a_n/\log n$ is equivalent to the convergence of $\sum_{k=1}^\infty S_k$. To show that the latter two series converge or diverge simultaneously, we will show that, except for a shift in the index k, the terms are of the same order of magnitude. More precisely, we will show that there exist positive constants c_1 and c_2 such that for all sufficiently large k,

$$T_k \begin{cases} \le c_1 S_{k-1}, \\ \ge c_2 S_{k+1}. \end{cases} \tag{1}$$

The proof of (1) depends on the fact that the intervals $(2^k, 2^{k+1}]$ contain the "proper share" of prime numbers, which in turn follows from the PNT. More specifically, the PNT implies $\pi(2^k) \sim 2^k/\log 2^k$ as $k \to \infty$ and hence

$$\begin{aligned}\pi(2^{k+1}) - \pi(2^k) &= \frac{2^{k+1}}{(k+1)\log 2} - \frac{2^k}{k\log 2} + o\left(\frac{2^k}{k}\right) \\ &= \frac{2^k}{k\log 2} - \frac{2^k}{k(k+1)\log 2)} + o\left(\frac{2^k}{k}\right) \\ &= \frac{2^k}{k\log 2} + o\left(\frac{2^k}{k}\right) \quad (k \to \infty).\end{aligned}$$

It follows that there exists k_0 such that for all $k \geq k_0$,

$$\pi(2^{k+1}) - \pi(2^k) \begin{cases} \leq \frac{2^{k+1}}{\log 2^k}, \\ \geq \frac{2^{k-1}}{\log 2^k}. \end{cases} \tag{2}$$

By (2) and the monotonicty of the coefficients a_n, we have for $k \geq k_0$

$$T_k \leq a_{2^k}\left(\pi(2^{k+1}) - \pi(2^k)\right) \leq a_{2^k}\frac{2^{k+1}}{\log 2^k},$$

while

$$S_{k-1} \geq a_{2^k}\frac{1}{\log 2^k}\#\{n : 2^{k-1} < n \leq 2^k\} = a_{2^k}\frac{2^{k-1}}{\log 2^k}.$$

Hence, for $k \geq k_0$, we have

$$T_k \leq 4S_{k-1},$$

which proves the upper bound in (1) with $c_1 = 4$.

To obtain the lower bound, we argue similarly, comparing the lower bound

$$T_k \geq a_{2^{k+1}}\frac{2^{k-1}}{\log 2^k}$$

with the upper bound

$$S_{k+1} \leq a_{2^{k+1}}\frac{2^{k+1}}{\log 2^{k+1}}.$$

This gives, for $k \geq k_0$,

$$T_k \geq S_{k+1} \geq \frac{1}{4}S_{k+1}$$

which proves the lower bound in (1) with $c_2 = 1/4$.

Problem 4

For positive integers k define the generalized von Mangoldt functions Λ_k by the identity $\sum_{d|n} \Lambda_k(d) = (\log n)^k$ (which for $k = 1$ reduces to the familiar identity for the ordinary von Mangoldt function $\Lambda(n)$). Show that $\Lambda_k(n) = 0$ if n has more than k distinct prime factors.

Solution

First solution. When $k = 1$, Λ_k is equal to the ordinary von Mangoldt function whose support is the set of prime powers, i.e., the set of integers having exactly one distinct prime factor. Thus, the assertion holds for $k = 1$. We prove the general case by induction on k.

To this end we establish the recursion formula

$$\Lambda_k = \Lambda_1 * \Lambda_{k-1} + \Lambda_{k-1} \log \quad (k \geq 2). \tag{1}$$

The claimed result easily follows from this identity by induction and the following observations: (i) multiplying an arithmetic function by another arithmetic function does not increase its support; (ii) the support of the convolution product of two functions f and g with support A and B, respectively, is obviously contained in the set $\{ab : a \in A, b \in B\}$. (Here, the support of an arithmetic functions is the set of positive integers at which the function is non-zero.) Thus, if we assume that $\Lambda_{k-1}(n) = 0$ if n has more than $k-1$ distinct prime factors, then $\Lambda_{k-1}(n) \log n$ has the same property, and $\Lambda_1 * \Lambda_{k-1}(n) = 0$ if n has more than k distinct prime factors, and so (1) shows that the assertion remains true for k.

To prove (1), it suffices to show that the convolution of the function on the right-hand side with 1 is equal to the function $\log^k$ (since $f * 1 = \log^k$ implies $f = \log^k * \mu = \Lambda_k * 1 * \mu = \Lambda_k$). We have, using the algebraic properties of the Dirichlet product,

$$\begin{aligned}
&(\Lambda_1 * \Lambda_{k-1} + \Lambda_{k-1} \log) * 1 \\
&\qquad = (\Lambda_1 * 1) * \Lambda_{k-1} + (\Lambda_{k-1} \log * 1) = \log * \Lambda_{k-1} + 1 * \Lambda_{k-1} \log .
\end{aligned}$$

The value of the latter function at a positive integer n is

$$\begin{aligned}
&\sum_{d|n} (\log d) \Lambda_{k-1}(n/d) + \sum_{d|n} \Lambda_{k-1}(n/d) \log(n/d) \\
&\qquad = (\log n) \sum_{d|n} \Lambda_{k-1}(n/d) = (\log n)(1 * \Lambda_{k-1})(n) = \log^k n,
\end{aligned}$$

which is what we wanted to show.

Second solution. We again use induction on k. The base case $k = 1$ follows from the known properties of the ordinary von Mangoldt function, so let $k \geq 1$ and assume the result has been proved for all indices $k' \leq k$. To prove the result for $k+1$, consider an integer n with at least $k+1$ distinct prime factors. We need to show that $\Lambda_{k+1}(n) = 0$.

In order to get the induction hypothesis into play, we choose a prime power p^m from the prime factorization of n and write n as $n = p^m n'$, so that $(p, n') = 1$ and n' has at least k distinct prime factors.

Now note that the squarefree divisors of n are exactly those of the form d' or $d'p$ with $d'|n'$, so we have

$$\begin{aligned}
\Lambda_{k+1}(n) &= \sum_{d|n} \mu(d) \log^{k+1}(n/d) \\
&= \sum_{d'|n'} \mu(d') \log^{k+1}(n'p^m/d') + \sum_{d'|n'} \mu(d'p) \log^{k+1}(n'p^{m-1}/d') \\
&= \sum_{d'|n'} \mu(d') \left(\log^{k+1}(n'p^m/d') - \log^{k+1}(n'p^{m-1}/d') \right).
\end{aligned}$$

Writing $\log(n'p^m/d) = \log(n'/d) + \log p^m$, $\log(n'p^{m-1}/d) = \log(n'/d) + \log p^{m-1}$, and expanding the $(k+1)$st powers of these logarithms by the binomial theorem, we see that

$$\log^{k+1}(n'p^m/d') - \log^{k+1}(n'p^{m-1}/d') = \sum_{i=0}^{k} a_i \log^i(n'/d'),$$

where the a_i are coefficients depending on p^m, but not on n' or d'. Hence,

$$\Lambda_{k+1}(n) = \sum_{i=0}^{k} a_i \sum_{d'|n'} \mu(d') \log^i(n'/d') = \sum_{i=0}^{k} a_i \Lambda_i(n'),$$

where $\Lambda_i = \log^i * \mu$ (with $\Lambda_0 = 1 * \mu = 1$). Since n' has at least k prime factors, we have $\Lambda_i(n') = 0$ for each $i \leq k$ by our induction hypothesis. (The term $\Lambda_0(n') = e(n')$ is also zero because n' is necessarily greater than 1 since it has at least $k(\geq 1)$ prime factors.) Thus we obtain $\Lambda_{k+1}(n) = 0$ as desired.

Problem 5

Call positive integer n round if it has no prime factors greater than $\sqrt{n}$. Let $R(x)$ denote the number of round integers $\le x$. Estimate $R(x)$ to within an error $O(x/\log x)$. (Hint: Estimate first the slightly different counting function

$$R_0(x) = \#\{n \le x : p|n \Rightarrow p \le \sqrt{x}\},$$

and then show that the difference between $R(x)$ and $R_0(x)$ is of order $O(x/\log x)$ and thus negligible.)

Solution

First, we have

$$\begin{aligned}
R_0(x) &= [x] - \#\{n \le x : \exists p, p|n, p > \sqrt{x}\} \\
&= x + O(1) - \sum_{\sqrt{x}<p\le x} [x/p] \\
&= x\left(1 - \sum_{\sqrt{x}<p\le x} \frac{1}{p}\right) + O(1) + O(\pi(x)) \\
&= x\left(1 - \log\log x + \log\log\sqrt{x}\right) + O\left(\frac{x}{\log\sqrt{x}}\right) \\
&= x(1-\log 2) + O\left(\frac{x}{\log x}\right).
\end{aligned}$$

Next note that, trivially, $R(x) \le R_0(x)$, and that $R_0(x) - R(x)$ is bounded above by the number, say $S(x)$, of positive integers $n \le x$ that have a prime divisor p with $\sqrt{n} < p \le \sqrt{x}$. Now,

$$\begin{aligned}
S(x) &\le \sum_{n\le x} \sum_{\substack{p|n \\ \sqrt{n}<p\le\sqrt{x}}} 1 \\
&= \sum_{p\le\sqrt{x}} \sum_{\substack{n<p^2 \\ p|n}} 1 \le \sum_{p\le\sqrt{x}} p \\
&\le \sqrt{x}\pi(\sqrt{x}) \ll \sqrt{x}\frac{\sqrt{x}}{\log\sqrt{x}} \ll \frac{x}{\log x},
\end{aligned}$$

by Chebyshev's estimate. Hence, $0 \le R_0(x) - R(x) = O(x/\log x)$, and therefore

$$R(x) = (1-\log 2)x + O\left(\frac{x}{\log x}\right).$$