Math 531, Fall 2005
Analytic Number Theory
Problem Set 5
Due November 7, 2005

Problem 1

Let $F(s) = \sum_{m,n=1}^{\infty} [m,n]^{-s}$. Determine the abscissa of convergence σ_c of $F(s)$ and express $F(s)$ in terms of the Riemann zeta function. (Hint: Express $F(s)$ as $\sum_{n=1}^{\infty} f(n) n^{-s}$, where $f(n) = \#\{(a,b) \in \mathbb{N}^2 : [a,b] = n\}$, and use the fact (established in an earlier homework problem) that f is multiplicative.)

Problem 2

Express the Dirichlet series $\sum_{n=1}^{\infty} d(n)^2 n^{-s}$ in terms of the Riemann zeta function. Then use this relation to derive a convolution identity relating the functions $d^2(n)$ and $d_4(n)$ (where $d_k(n)\#\{(a_1,\dots,a_k) \in \mathbb{N}^k : a_1 \dots a_k = n\}$ is the generalized divisor function).

Problem 3

Evaluate the series $\sum_{(m_1,\dots,m_r)=1} m_1^{-s} \cdots m_r^{-s}$, where the summation is over all tuples $(m_1,\dots,m_r)$ of positive integers that are relatively prime, in terms of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

Problem 4

Without using the PNT (you may use Chebyshev's estimates or Mertens' estimates), obtain an asymptotic estimate for the partial sums

$$S(x) = \sum_{p \le x} \frac{1}{p \log p}$$

(with as as good an error term as you can get using only results at the level of Chebyshev or Mertens).

Problem 5

Let $Q(x) = \prod_{p\le x}(1+1/p)$. Obtain an estimate for $Q(x)$ with relative error $O(1/\log x)$. Express the constant arising in this estimate in terms of well-known mathematical constants. (Hint: Relate $Q(x)$ to the product $P(x) = \prod_{p\le x}(1-1/p)$ estimated by Mertens' formula.)

Problem 6

Let $\lambda > 1$ and $t \neq 0$ be fixed real numbers, and $S_{t,\lambda}(x) = \sum_{x<n\le\lambda x} n^{-1-it}$. Obtain an estimate for $S_{t,\lambda}(x)$ as $x \to \infty$ with error term $O_{t,\lambda}(1/x)$. Deduce from this estimate that for any non-zero t and any $\lambda > 1$, the limit $\lim_{x\to\infty} |S_{t,\lambda}(x)|$ exists, and that, for given $t \neq 0$ and *suitable* choices of λ this limit is non-zero. (Thus, by Cauchy's criterion, the series $\sum_{n=1}^{\infty} n^{-1-it}$ diverges for every real $t \neq 0$.)

Problem 7*

Use the PNT (or a result equivalent to the PNT) to obtain the estimate

$$\sum_{n\le x}\mu^2(n) = \frac{6}{\pi^2}x + o(\sqrt{x}) \quad (x\to\infty). \tag{0}$$

(Hint: This was proved in class with the weaker error term $O(\sqrt{x})$, without making use of the PNT. To get the desired estimate, look at the term that generated the $O(\sqrt{x})$ error, and try to estimate it more carefully, using the PNT.)

Problem 8*

(Challenge/bonus problem—deadline 11/28/05) Let $f = 1 * g$. Wintner's theorem shows that if the series

$$\sum_{n=1}^{\infty}\frac{g(n)}{n} \tag{1}$$

converges **absolutely**, then the mean value $M(f)$ of f exists and is equal to the sum of the series (1).

(i) Show that the conclusion of Wintner's theorem remains valid if the series (1) converges only conditionally and if, in addition,

$$\limsup_{x\to\infty} \frac{1}{x} \sum_{n\le x} |g(n)| < \infty. \tag{2}$$

(ii) Show that condition (2) cannot be dropped; i.e., construct an example of a function g for which the series (1) converges, but the function $f = 1 * g$ does not have a mean value.

Additional problems from Apostol's text (not to be turned in)

Here are some recommended additional practice problems from Chapter 11 in Apostol. Most of the problems in this chapter are on the easy/routine side.

Problem 2: (a) and (c) are routine exercises in partial summation. Part (b) is a bit harder and requires a careful $\epsilon - x_0(\epsilon)$ argument; for that reason, it's quite instructive and a good problem to practice "epsilonics"!

Problems 4–10: These are exercises in manipulating Dirichlet series and Euler products and using identities for arithmetic functions, and make for good practice problems and warm-up exercises for the first three problems of the current assignment. To gain the most out of these problems, try to establish the identities given in these problems *without referring to the right-hand side*, i.e., simply try to express the series on the left in terms of the Riemann zeta function.

Problems 11–15: These problems deal with more general identities and relations of this type, but are not much harder. The second part of 15 is part of the current assignment, in a slightly weaker form.