Math 531, Fall 2005
Analytic Number Theory
Problem Set 5
Solutions

Problem 1

Let $F(s)=\sum_{m,n=1}^{\infty}[m,n]^{-s}$. Determine the abscissa of convergence σ_c of $F(s)$ and express $F(s)$ in terms of the Riemann zeta function. (Hint: Express $F(s)$ as $\sum_{n=1}^{\infty} f(n)n^{-s}$, where $f(n)=\#\{(a,b)\in\mathbb{N}^2:[a,b]=n\}$, and use the fact (established in an earlier homework problem) that f is multiplicative.)

Solution

We have $F(s)=\sum_{n=1}^{\infty} f(n)n^{-s}$, where $f(n)=\#\{(a,b)\in\mathbb{N}^2:[a,b]=n\}$. Since $f(n)\ge 1$, $F(s)$ diverges at $s=1$, and so $\sigma_c\ge 1$. On the other hand, $F(s)$ converges absolutely for $\sigma>1$, since, for any prime power p^m, $f(p^m)=2m+1\le C_\epsilon 2^{m\epsilon}\le C_\epsilon p^{m\epsilon}$ with a suitable constant C_ϵ, and so $\sum_{p^m}|f(p^m)|p^{-m\sigma}<\infty$ for $\sigma>1$. Hence $\sigma_a\le 1$, and therefore $\sigma_c=\sigma_a=1$.

To evaluate $F(s)$, write $F(s)$ as an Euler product $F(s)=\prod_p F_p(s)$ with

$$\begin{aligned} F_p(s)&=\sum_{m=0}^{\infty}\frac{f(p^m)}{p^{ms}}=\sum_{m=0}^{\infty}\frac{2m+1}{p^{ms}}\\ &=2\cdot\frac{p^{-s}}{(1-p^{-s})^2}+\frac{1}{1-p^{-s}}\\ &=\frac{1+p^{-s}}{(1-p^{-s})^2}=\frac{1-p^{-2s}}{(1-p^{-s})^3}. \end{aligned}$$

The right-hand side here can be recognized as the p-th factor of the Euler product for $\zeta(2s)^{-1}$, multiplied by the p-th factor of the Euler product for $\zeta(s)^3$. Hence

$$F(s)=\prod_p\frac{1-p^{-2s}}{(1-p^{-s})^3}=\zeta(s)^3\zeta(2s)^{-1}.$$

Problem 2

Express the Dirichlet series $\sum_{n=1}^{\infty} d(n)^2 n^{-s}$ in terms of the Riemann zeta function. Then use this relation to derive a convolution identity relating the functions $d^2(n)$ and $d_4(n)$ (where $d_k(n)\#\{(a_1,\dots,a_k)\in\mathbb{N}^k : a_1\dots a_k = n\}$ is the generalized divisor function).

Solution

Let $f(n)=d(n)^2$. Since $d(n)\ll_\epsilon n^\epsilon$ for every $\epsilon>0$, the given Dirichlet series is absolutely convergent in the half-plane $\sigma>1$, and there has the Euler product representation

$$F(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\prod_p F_p(s)$$

with

$$F_p(s)=\sum_{m=0}^{\infty}\frac{d(p^m)^2}{p^{ms}}=\sum_{m=0}^{\infty}\frac{(m+1)^2}{p^{ms}}.$$

Using the identity

$$\sum_{m=0}^{\infty}(m+1)^2x^m=(1+x)(1-x)^{-3}=(1-x^2)(1-x)^{-4}\quad(|x|<1),$$

(which is easily established, starting from the identity $\sum_{n=0}^{\infty}x^n=(1-x)^{-1}$, differentiating, then multiplying with x, and differentiating a second time), we see that $F_p(s)=(1-p^{-2s})(1-p^{-s})^{-4}$. Hence

$$F(s)=\prod_p(1-p^{-2s})\prod_p(1-p^{-s})^{-4}=\zeta(2s)^{-1}\zeta(s)^4, \tag{1}$$

or, equivalently,

$$\zeta(s)^4=\zeta(2s)F(s). \tag{2}$$

Since $\zeta(2s)$ is the Dirichlet series associated with the characteristic function s of the squares, and $\zeta(s)^4$ the Dirichlet series of $1*1*1*1=d_4$, the identity (2) is equivalent to the convolution identity $d_4=s*f=s*d_2^2$, i.e., we have

$$d_4(n)=\sum_{m^2|n}d(n/m^2)^2\quad(n\in\mathbb{N}).$$

Alternatively, from the identity (1) we obtain the identity $d^2=g*d_4$ with g defined by $g(m^2)=\mu(m)$ and $g(n)=0$ if n is not a square, i.e.,

$$d(n)^2=\sum_{m^2|n}\mu(m)d_4(n/m^2)\quad(n\in\mathbb{N}).$$

Problem 3

Evaluate the series $\sum_{(m_1,\dots,m_r)=1} m_1^{-s}\cdots m_r^{-s}$, where the summation is over all tuples $(m_1,\dots,m_r)$ of positive integers that are relatively prime, in terms of the Riemann zeta function $\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$.

Solution

Let $F(s)$ denote the series in question. We will show that $F(s)=\zeta(s)^{2r}/\zeta(sr)$, for $\sigma>1$. A two line proof, which only requires a basic property of the gcd (namely the fact that $(m_1,\dots,m_r)=d$ if any only if $d|n_i$ for all i and the numbers $n_i=m_i/d$, $i=1,\dots,r$, are relatively prime), but no knowledge from the theory of arithmetic functions, runs as follows:

$$\begin{aligned}\zeta(s)^r &= \sum_{m_1,\dots,m_r=1}^{\infty}\frac{1}{m_1^s\cdots m_r^s}=\sum_{d=1}^{\infty}\sum_{(m_1,\dots,m_r)=d}^{\infty}\frac{1}{m_1^s\cdots m_r^s}\\ &=\sum_{d=1}^{\infty}\frac{1}{d^{sr}}\sum_{(n_1,\dots,n_r)=1}\frac{1}{n_1^s\dots n_r^s}=\zeta(sr)F(s),\end{aligned}$$

where the inner sums are sums over all r-tupels $(m_1,\dots,m_r)$ (resp. $(n_1,\dots,n_r)$) of positive integers satisfying the given conditions.

[A slightly longer proof can be given by using the fact that $\sum_{d|(m_1,\dots,m_r)}\mu(d)=e((m_1,\dots,m_r))=\sum_{d|m_1,\dots d|m_r}\mu(d)$ to eliminate the coprimality condition in the summation.]

Problem 4

Without using the PNT (you may use Chebyshev's estimates or Mertens' estimates), obtain an asymptotic estimate for the partial sums

$$S(x) = \sum_{p \le x} \frac{1}{p \log p}$$

(with as good an error term as you can get using only results at the level of Chebyshev or Mertens).

Solution

A natural approach to this problem would be to use partial summation to remove the "weight" $1/(p \log p)$ in the sum $S(x)$, but this would lead to an expression for $S(x)$ in terms of the prime counting function $\pi(x)$ and require the PNT in order to obtain good estimates for $S(x)$. We can avoid the PNT by only removing the factor $1/\log p$ in the sum $S(x)$, thus relating this sum to the sums

$$T(x) = \sum_{p \le x} \frac{1}{p}.$$

The result is

$$S(x) = \frac{T(x)}{\log x} + \int_2^x \frac{T(y)}{y \log^2 y} dy \quad (x \ge 2). \tag{1}$$

Substituting Mertens' estimate

$$T(x) = \log\log x + C + O\left(\frac{1}{\log x}\right) \quad (x \ge 2),$$

where C is a constant, we obtain

$$S(x) = \frac{\log\log x}{\log x} + \frac{C}{\log x} + O\left(\frac{1}{\log^2 x}\right) + I(x), \tag{2}$$

where $I(x)$ denotes the integral on the right of (1). Since $T(x) \ll \log\log x$ for large x and

$$\int_2^\infty \frac{|\log\log y|}{y \log^2 y} dy = \int_{\log 2}^\infty \frac{|\log u|}{u^2} du < \infty,$$

the integral $I(x)$ converges absolutely, when extended to infinity, and we have

$$I(x) = \int_2^\infty \frac{T(y)}{y \log^2 y} dy - \int_x^\infty \frac{T(y)}{y \log^2 y} dy = I - R(x), \tag{3}$$

say, where I is a constant (equal to the value of the infinite integral). Using again Mertens' estimate, we get

$$
\begin{aligned}
R(x) &= \int_x^\infty \frac{\log\log y + C + O(1/\log y)}{y(\log y)^2}\,dy \\
&= \int_{\log x}^\infty \frac{\log u + C + O(1/u)}{u^2}\,du \\
&= \int_{\log x}^\infty \frac{\log u}{u^2}\,du + C\int_{\log x}^\infty \frac{1}{u^2}\,du + O\left(\int_{\log x}^\infty \frac{1}{u^3}\,du\right) \\
&= J(\log x) + \frac{C}{\log x} + O\left(\frac{1}{\log^2 x}\right),
\end{aligned}
\tag{4}
$$

where

$$
J(t) = \int_t^\infty \frac{\log u}{u^2}\,du = \frac{\log t}{t} + \int_t^\infty \frac{1}{u^2}\,dt = \frac{\log t}{t} + \frac{1}{t}. \tag{5}
$$

Combining (2)–(5) we obtain

$$
S(x) = I - \frac{1}{\log x} + O\left(\frac{1}{\log^2 x}\right),
$$

where I is defined by (3). This is the desired estimate for $S(x)$, and it is the best estimate one can get without using the PNT. In particular, this estimate shows that the infinite series $\sum_p 1/(p\log p)$ converges, in contrast to the series $\sum_{n=1}^\infty 1/(n\log n)$, which diverges.

Problem 5

Let $Q(x) = \prod_{p \le x}(1 + 1/p)$. Obtain an estimate for $Q(x)$ with relative error $O(1/\log x)$. Express the constant arising in this estimate in terms of well-known mathematical constants. (Hint: Relate $Q(x)$ to the product $P(x) = \prod_{p \le x}(1 - 1/p)$ estimated by Mertens' formula.)

Solution

With $P(x)$ and $Q(x)$ as given in the problem, we have

$$P(x)Q(x) = \prod_{p \le x} \left(1 - \frac{1}{p^2}\right) = \prod_{p} \left(1 - \frac{1}{p^2}\right) R(x), \tag{1}$$

where $R(x) = \prod_{p > x}(1 - 1/p^2)^{-1}$. The infinite product here is equal to

$$\prod_{p} \left(1 - \frac{1}{p^2}\right) = \zeta(2)^{-1} = \frac{6}{\pi^2}, \tag{2}$$

and the "tail product" $R(x)$ can be estimated by

$$\begin{aligned} R(x) &= \exp\left\{ - \sum_{p > x} \log\left(1 - \frac{1}{p^2}\right)\right\} = \exp\left\{ O\left(\sum_{p > x} \frac{1}{p^2}\right)\right\} \\ &= \exp\left\{ O\left(\frac{1}{x}\right)\right\} = 1 + O\left(\frac{1}{x}\right). \end{aligned} \tag{3}$$

Moreover, by Mertens' formula we have

$$P(x) = \frac{e^{-\gamma}}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right), \tag{4}$$

Combining (1)–(4) we obtain

$$\begin{aligned} Q(x) &= P(x)^{-1} \frac{6}{\pi^2} \left(1 + O\left(\frac{1}{x}\right)\right) \\ &= \frac{6e^{\gamma}}{\pi^2}(\log x)\left(1 + O\left(\frac{1}{\log x}\right)\right), \end{aligned}$$

which is an estimate of the required form.

Problem 6

Let $\lambda > 1$ and $t \neq 0$ be fixed real numbers, and $S_{t,\lambda}(x) = \sum_{x<n\leq \lambda x} n^{-1-it}$. Obtain an estimate for $S_{t,\lambda}(x)$ as $x \to \infty$ with error term $O_{t,\lambda}(1/x)$. Deduce from this estimate that for any non-zero t and any $\lambda > 1$, the limit $\lim_{x\to\infty} |S_{t,\lambda}(x)|$ exists, and that, for given $t \neq 0$ and *suitable* choices of λ, this limit is non-zero. (Thus, by Cauchy's criterion, the series $\sum_{n=1}^{\infty} n^{-1-it}$ diverges for every real $t \neq 0$.)

Solution

Using Euler's summation formula in its crudest form, namely

$$\sum_{y<n\leq x} f(n) = \int_y^x f(u)du + O(|f(y)|) + O(|f([x])|) + O\left(\int_y^x |f'(u)|du\right),$$

we get, in the case $t \neq 0$,

$$\begin{aligned} S_{t,\lambda}(x) &= \int_x^{\lambda x} u^{-1-it}du + O\left(\frac{1}{x}\right) + O\left(\int_x^{\lambda x} |1+it|u^{-2}du\right) \\ &= \frac{(\lambda x)^{-it} - x^{-it}}{-it} + O_t(x^{-1}) = C_{t,\lambda}x^{-it} + O_t(x^{-1}), \end{aligned}$$

where $C_{t,\lambda} = (\lambda^{-it} - 1)/(-it)$. Hence

$$|S_{t,\lambda}(x)| = |C_{t,\lambda}x^{-it} + O_t(x^{-1})| = |C_{t,\lambda}| + O(x^{-1}),$$

so the limit $\lim_{x\to\infty} |S_{t,\lambda}(x)|$ exists and is equal to $|C_{t,\lambda}|$. Moreover, if λ is such that $\lambda^{-it} \neq 1$ (i.e., if $t \log \lambda$ is not a multiple of 2π), then $C_{t,\lambda} \neq 0$, and so $S_{t,\lambda}(x)$ does not tend to 0 as $x \to \infty$.

Problem 7*

Use the PNT (or a result equivalent to the PNT) to obtain the estimate

$$\sum_{n\le x}\mu^2(n)=\frac{6}{\pi^2}x+o(\sqrt{x})\quad (x\to\infty). \tag{0}$$

(Hint: This was proved in class with the weaker error term $O(\sqrt{x})$, without making use of the PNT. To get the desired estimate, look at the term that generated the $O(\sqrt{x})$ error, and try to estimate it more carefully, using the PNT.)

Solution

Defining g by $\mu^2=g*1$, we have (from class, or by a straightforward computation)

$$g(n)=\begin{cases}\mu(m) & \text{if } n=m^2,\\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

Let $S(x)$ denote the sum to be estimated, and let $M(x)=\sum_{n\le x}\mu(n)$. For any $x\ge 1$ and $y\in[1,x]$ we have, by the Dirichlet hyperbola method,

$$S(x)=\sum_{n\le x}\sum_{d|n}g(d)=\sum_{\substack{d,m\le x\\ dm\le x}}g(d)=S_1(x,y)+S_2(x,y)-S_3(x,y),$$

where

$$\begin{aligned}
S_1(x,y)&=\sum_{d\le y}g(d)\left[\frac{x}{d}\right]=\sum_{m\le\sqrt{y}}\mu(m)\left[\frac{x}{m^2}\right]\\
S_2(x,y)&=\sum_{n\le x/y}\sum_{d\le x/n}g(d)=\sum_{n\le x/y}M(\sqrt{x/n}),\\
S_3(x,y)&=\sum_{d\le y}\sum_{n\le x/y}g(d)=M(\sqrt{y})\left[\frac{x}{y}\right].
\end{aligned}$$

To estimate the sums $S_i(x,y)$ we introduce the function

$$\epsilon(x)=\sup_{y\ge x}\frac{|M(y)|}{y},$$

and exploit the fact that $\epsilon(x)$ tends to zero as $x\to\infty$ (which is just a restatement of the relation $M(x)=o(x)$). This gives immediately the bounds

$$|S_3(x,y)|\le\epsilon(\sqrt{y})\sqrt{y}\cdot\frac{x}{y}=\frac{\epsilon(\sqrt{y})}{\sqrt{y}}x \tag{2}$$

and

$$\begin{aligned} |S_2(x,y)| &\le \epsilon(\sqrt{y}) \sum_{n \le x/y} \sqrt{\frac{x}{n}} \\ &\ll \epsilon(\sqrt{y})\sqrt{x} \cdot \sqrt{\frac{x}{y}} = \frac{\epsilon(\sqrt{y})}{y} x, \end{aligned} \tag{3}$$

since $\sum_{n \le x} n^{-1/2} \ll x^{1/2}$ for $x \ge 1$. Furthermore, we have

$$\begin{aligned} S_1(x,y) &= x \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} - x \sum_{m > \sqrt{y}} \frac{\mu(m)}{m^2} + O\left(\sum_{m \le \sqrt{y}} 1 \right) \\ &= x\zeta(2)^{-2} - xR(\sqrt{y}) + O(\sqrt{y}) = \frac{\pi^2}{6} x - xR(\sqrt{y}) + O(\sqrt{y}), \end{aligned} \tag{4}$$

where

$$R(x) = \sum_{n > x} \frac{\mu(n)}{n^2}.$$

Partial summation gives

$$\begin{aligned} |R(x)| &\le \frac{|M(x)|}{x^2} + 2\int_x^{\infty} \frac{|M(t)|}{t^3} \\ &\le \epsilon(x) \left(\frac{1}{x} + 2\int_x^{\infty} \frac{dt}{t^2} \right) = \frac{3\epsilon(x)}{x} \end{aligned}$$

Substituting this bound into (4) we obtain

$$S_1(x,y) = \frac{6}{\pi^2} x + O\left(\frac{\epsilon(\sqrt{y})}{\sqrt{y}} x \right) + O(\sqrt{y}). \tag{5}$$

Combining (2)–(5) gives

$$S(x) = \frac{6}{\pi^2} x + O\left(\frac{\epsilon(\sqrt{y})}{\sqrt{y}} x \right) + O(\sqrt{y}).$$

We apply this estimate with $y = \delta x$, where $\delta \in (0,1)$ is fixed and $x \ge 1/\delta$, to get

$$S(x) = \frac{6}{\pi^2} x + O\left(\frac{\epsilon(\sqrt{\delta x})}{\delta} \sqrt{x} \right) + O(\sqrt{\delta x})). \tag{6}$$

Since $\epsilon(\sqrt{\delta x}) = o(1)$ as $x \to \infty$, this implies that

$$\limsup_{x \to \infty} \frac{|S(x) - (6/\pi^2)x|}{\sqrt{x}} \le c\sqrt{\delta},$$

for a suitable absolute constant c. Letting now $\delta \to 0$, we obtain the desired relation.

Remark. Using the Dirichlet hyperbola method (or an argument equivalent to it) was essential to getting $o(\sqrt{x})$ as error term. In particular, a direct estimation of $\sum_{m\le\sqrt{x}}\mu(m)[x/m^2]$ (this corresponds to the sum $S_1(x,y)$ above with $y=x$) would give a term $x\sum_{m\le\sqrt{x}}\mu(m)m^{-2}$, which is easy to handle, but also introduce a term $\sum_{m\le\sqrt{x}}\mu(m)\{x/m^2\}$, for which a trivial estimate (namely by $\sum_{m\le\sqrt{x}}\{x/m^2\}$, would only give a bound $O(\sqrt{x})$, which is not good enough. To successfully estimate the latter term, one would have to essentially work backwards, writing $\{x/m^2\}$ as $x/m^2-[x/m^2]$, and then apply the above argument, breaking the sum up according to the hyperbola method.

Problem 8*

(Challenge/bonus problem—deadline 11/28/05) Let $f = 1 * g$. Wintner's theorem shows that if the series

$$\sum_{n=1}^{\infty} \frac{g(n)}{n} \tag{1}$$

converges **absolutely**, then the mean value $M(f)$ of f exists and is equal to the sum of the series (1).

(i) Show that the conclusion of Wintner's theorem remains valid if the series (1) converges only conditionally and if, in addition,

$$\limsup_{x\to\infty} \frac{1}{x} \sum_{n\le x} |g(n)| < \infty. \tag{2}$$

(ii) Show that condition (2) cannot be dropped; i.e., construct an example of a function g for which the series (1) converges, but the function $f = 1 * g$ does not have a mean value.

Solution

[Deferred till after the deadline.]