

Math 531, Fall 2005  
Analytic Number Theory  
Problem Set 6  
Due November 18, 2005

For the following problems, you can use the PNT with the error term obtained in class, i.e.,  $\psi(x) = x + O(x \exp(-c(\log x)^\alpha))$ , where  $c > 0$  is a positive constant and  $\alpha = 1/10$  (though the numerical value of  $\alpha$  does not matter), or the corresponding estimates for  $\theta(x)$  and  $\pi(x)$ . In most cases, a weaker form, in which the relative error is term  $O_k((\log x)^{-k})$  for some fixed constant  $k$  is sufficient. (The latter form has the advantage of being easier to work with than versions with exponential error term.)

**Problem 1**

Show that, if  $x$  is sufficiently large, then interval  $[2, x]$  contains more primes than the interval  $(x, 2x]$ .

**Problem 2**

Define  $A(x)$  by  $\pi(x) = x/(\log x - A(x))$ . Show that  $A(x) = 1 + O(1/\log x)$  for  $x \geq 2$ .

**Remark.** This result is of historical interest for the following reason: While the function  $x/\log x$  is asymptotically equal to  $\pi(x)$  by the prime number theorem, examination of numerical data suggests that the function  $x/\log x$  is not a particularly good approximation to  $\pi(x)$ . Therefore, in the early (pre-PNT) history of prime number theory several other functions were suggested as suitable approximations to  $\pi(x)$ . In particular, Legendre proposed the function  $x/(\log x - 1.08366)$  (The particular value of the constant 1.08366 was presumably obtained by some kind of regression analysis on the data.) On the other hand, Gauss suggested that  $x/(\log x - 1)$  was a better match to  $\pi(x)$ . The problem settles this dispute, showing that Gauss had it right.

### Problem 3

Let  $f(n) = \Lambda(n) - 1$ . Show that the Dirichlet series  $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$  converges for every  $s$  on the line  $\sigma = 1$ , and obtain an estimate for the rate of convergence, i.e., the difference  $F(s) - \sum_{n \leq x} f(n)n^{-s}$ , when  $s = 1 + it$  for some fixed  $t$ . (The estimate may depend on  $t$ , but try to get as good an error term as possible assuming the PNT with exponential error term.)

### Problem 4

Evaluate the integral

$$I_k(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^s}{s^k} ds,$$

where  $k$  is an integer  $\geq 2$ , and  $y$  and  $c$  are positive real numbers. Then use this evaluation to derive a Perron type formula, analogous to the two formulas proved in class, for the integral

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s^k} ds,$$

where  $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$  is a Dirichlet series, stating any conditions that are needed for this formula to be valid.

### Problem 5

Show that if  $\zeta(s)$  has no zeros in  $\sigma > 1/2$  (i.e., if the Riemann Hypothesis holds) and, in addition, satisfies

$$(*) \quad \left| \frac{1}{\zeta(s)} \right| \ll_{\epsilon, \sigma_0} t^\epsilon \quad (|t| \geq 1, \sigma \geq \sigma_0)$$

for any fixed  $\epsilon > 0$  and  $\sigma_0 > 1/2$ , then

$$(1) \quad \sum_{n \leq x} \mu(n) = O_\epsilon(x^{1/2+\epsilon})$$

holds for any fixed  $\epsilon > 0$ . (Hint: Use Perron's formula for  $M(\mu, x)$  (rather than the "smoothed" version of Perron's formula for  $M_1(\mu, x)$ ). This is because the transition from  $M_1$  to  $M$  introduces an additional error term that is larger than the square-root type error allowed in (1).)

**Remarks:** It can be shown (though this is beyond the scope of this class) that the Riemann Hypothesis implies (\*), and hence (1). In class it was shown that the converse also holds, i.e., (1) implies the Riemann Hypothesis. Thus, the Riemann Hypothesis is, in fact, equivalent to (1). In much the same way, one can show that the Riemann Hypothesis is equivalent to the assertion that the PNT holds in the form  $\psi(x) = x + O_\epsilon(x^{1/2+\epsilon})$ .

## Additional practice problems from Apostol's text

Here are some comments on the problems in Chapter 13 of Apostol's text.

Problem 1: This was proved in class using an approach similar to the one suggested here. The argument is a routine  $\epsilon$ - $x_0(\epsilon)$  argument, and worth working out on your own.

Problem 2: A very easy estimation problem, requiring only Chebyshev's estimates. The function  $a(n)$  came up in class in connection with the proof of Mertens' formula (it was denoted by  $\kappa(n)$  there).

Problem 3: (a) is a special case of the general Perron formula (the one for  $M(f, x)$ ) proved in class. (b) is an easy consequence of (a).

Problem 4: This was proved in class. The argument is not hard, and worth revisiting.

Problem 5: An instructive exercise in  $\epsilon$ - $x_0(\epsilon)$  reasoning.

Problems 6/7: These are, in a slightly more general form, part of the current assignment.

Problems 8–10: These problems depend on material (Dirichlet  $L$ -series) that we haven't covered yet.

Problem 11: Easy!

Problems 12/13: The first came up in an earlier assignment. The second is just an application of the first.