Math 531, Fall 2005
Analytic Number Theory
Problem Set 6
Solutions

Problem 1

Show that, if x is sufficiently large, then interval $[2, x]$ contains more primes than the interval $(x, 2x]$.

Solution

We need to show that $\pi(x) > \pi(2x) - \pi(x)$, or equivalently $\pi(x) - (\pi(2x) - \pi(x)) > 0$, holds for all sufficiently large x. By the prime number theorem in the form $\pi(x) = \mathrm{Li}(x) + O_k(x(\log x)^{-k})$, we have

$$\pi(x) - (\pi(2x) - \pi(x)) = \mathrm{Li}(x) - (\mathrm{Li}(2x) - \mathrm{Li}(x)) + O_k\left(\frac{x}{\log^k x}\right). \tag{1}$$

We will show that, for x sufficiently large,

$$\mathrm{Li}(x) - (\mathrm{Li}(2x) - \mathrm{Li}(x)) \gg \frac{x}{\log^2 x}. \tag{2}$$

This, together with (1) with $k = 3$, implies that the right-hand side of (1) is positive when x is sufficiently large.

A slick proof of (2), which uses only the definition $\mathrm{Li}(x) = \int_2^x (\log t)^{-1} dt$, goes as follows: For $x \geq 4$ we have

$$\begin{aligned}\mathrm{Li}(x) - (\mathrm{Li}(2x) - \mathrm{Li}(x)) &= \int_2^x \frac{1}{\log t} dt - \int_x^{2x} \frac{1}{\log t} dt \\ &= \int_2^x \left(\frac{1}{\log t} - \frac{1}{\log(t+x)}\right) dt + O(1),\end{aligned}$$

where the term $O(1)$ comes from the contribution of the range $x \le t \le x+2$. The integrand in the last integral is nonnegative for all $t \in [2, x]$, and for $2 \le t \le x/2$ satisfies

$$\frac{1}{\log t} - \frac{1}{\log(t+x)} = \frac{\log(1+x/t)}{(\log t)(\log(t+x))} \ge \frac{\log 2}{\log^2(2x)}.$$

Hence this integral is $\gg x/\log^2 x$ for $x \ge 4$, and (2) follows.

Alternatively, (2) can be derived from the expansion (which was proved in class)

$$\mathrm{Li}(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right). \tag{3}$$

Replacing here x by $2x$, we get

$$\begin{aligned} \mathrm{Li}(2x) &= \frac{2x}{\log x + \log 2} + \frac{2x}{(\log x + \log 2)^2} + O\left(\frac{2x}{(\log x)^3}\right) \\ &= \frac{2x}{\log x}\left(1 - \frac{\log 2}{\log x}\right) + \frac{2x}{(\log x)^2} + O\left(\frac{2x}{(\log x)^3}\right). \end{aligned} \tag{4}$$

From (3) and (4) we obtain

$$2\,\mathrm{Li}(x) - \mathrm{Li}(2x) = \frac{2(\log 2)x}{(\log x)^2} + O\left(\frac{2x}{(\log x)^3}\right),$$

which implies (2).

Problem 2

Define $A(x)$ by $\pi(x) = x/(\log x - A(x))$. Show that $A(x) = 1 + O(1/\log x)$ for $x \geq 2$.

Solution

Solving for $A(x)$, we obtain $A(x) = \log x - (x/\pi(x))$. Using the PNT in the form $\pi(x) = Li(x) + O(x(\log x)^{-3}))$ along with the expansion

$$Li(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right)$$

we get

$$\pi(x) = \frac{x}{\log x}\left(1 + \frac{1}{\log x}\right)\left(1 + O\left(\frac{1}{(\log x)^2}\right)\right),$$

which implies

$$\begin{aligned}
\frac{x}{\pi(x)} &= \frac{\log x}{(1 + (\log x)^{-1})(1 + O((\log x)^{-2}))} \\
&= (\log x)\left(1 - \frac{1}{\log x}\right) + O\left(\frac{1}{(\log x)^2}\right) = \log x - 1 + O\left(\frac{1}{(\log x)^2}\right).
\end{aligned}$$

Hence, $A(x) = \log x - (x/\pi(x)) = 1 + O(1/\log x)$, as claimed.

Problem 3

Let $f(n) = \Lambda(n) - 1$. Show that the Dirichlet series $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ converges for every s on the line $\sigma = 1$, and obtain an estimate for the rate of convergence, i.e., the difference $F(s) - \sum_{n\le x} f(n)n^{-s}$, when $s = 1 + it$ for some fixed t. (The estimate may depend on t, but try to get as good an error term as possible assuming the PNT with exponential error term.)

Solution

Let $S(x) = \sum_{n\le x} f(n)$ and $F_x(s) = \sum_{n\le x} f(n)n^{-s}$. By partial summation, we have, for any $x \ge 1$ and any complex number s,

$$F_x(s) = S(x)x^{-s} + s\int_1^x S(u)u^{-s-1}du. \tag{1}$$

To show that the series converges for $\sigma = 1$, it suffices to use the PNT in the form $\psi(x) = x + O(x/\log^2 x)$. Since $S(x) = \psi(x) - [x]$, we have $S(x) = O(x/(\log x)^2)$. Hence, for $\sigma = 1$, the first term on the right of (1) tends to 0 if $x \to \infty$, while the integral is convergent if extended to infinity, since $|S(u)u^{-s-1}| \ll u^{-1}(\log u)^{-2}$ for $u \ge 2$ and the integral $\int_2^\infty u^{-1}(\log u)^{-2}du$ converges. Thus the right hand side of (1) tends to a finite limit as $x \to \infty$. This proves the convergence of the series $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ for $\sigma = 1$, and it also shows that $F(s)$ has the integral representation

$$F(s) = s\int_1^\infty S(u)u^{-s-1}du. \tag{2}$$

To obtain an estimate for the tails of the series $F(s)$, i.e., $F(s) - F_x(s)$, we use (1) and (2) we get, for $\sigma = 1$,

$$F(s) - F_x(s) \ll |S(x)|x^{-1} + |s|\int_x^\infty |S(u)|u^{-2}du. \tag{3}$$

We now estimate the right-hand side using the PNT in the form

$$\psi(x) = x + O\left(x\exp\left\{-c(\log x)^\alpha\right\}\right) \quad (x \ge 2), \tag{4}$$

where c and α are positive constants. (In class we proved this with $\alpha = 1/10$, though the specific value of α does not affect the argument below.) Since $S(x) = \psi(x) - [x]$ and

$$\begin{aligned}\exp\left\{-c(\log x)^\alpha\right\} &= \exp\left\{-(c/2)(\log x)^\alpha\right\}\cdot\exp\left\{-(c/2)(\log x)^\alpha\right\}\\ &\ll \frac{1}{(\log x)^2}\exp\left\{-(c/2)(\log x)^\alpha\right\},\end{aligned}$$

we see that

$$S(u) \ll \frac{u}{(\log u)^2}\exp\left\{-(c/2)(\log x)^\alpha\right\} \quad (u \ge x).$$

Substituting this estimate into (3) gives, for $\sigma = 1$,

$$\begin{aligned} F(s) - F_x(s) &\ll \exp\{-(c/2)(\log x)^\alpha\} \left(\frac{1}{(\log x)^2} + |s| \int_x^\infty \frac{1}{(\log u)^2} du \right) \\ &\ll \exp\{-(c/2)(\log x)^\alpha\} \left(\frac{1}{(\log x)^2} + \frac{|s|}{\log x} \right) \ll_s \exp\{-(c/2)(\log x)^\alpha\}. \end{aligned}$$

Thus we have for the tail $F(s) - F_x(s)$ a bound of the same type as the relative error term in the prime number theorem (4), except for a smaller value of the constant c.

Problem 4

Evaluate the integral

$$I_k(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s^k} ds,$$

where k is an integer ≥ 2, and y and c are positive real numbers. Then use this evaluation to derive a Perron type formula, analogous to the two formulas proved in class, for the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\frac{x^s}{s^k} ds,$$

where $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ is a Dirichlet series, stating any conditions that are needed for this formula to be valid.

Solution

We will show that

$$(1) \qquad I_k(y) = \begin{cases} (\log y)^{k-1}/(k-1)! & \text{if } y \geq 1, \\ 0 & \text{if } 0 < y < 1. \end{cases}$$

If $y > 1$, we replace the path of integration $[c-iT, c+iT]$ by a path consisting of the horizontal segments $[a \pm iT, c \pm iT]$ and the vertical segment $[a-iT, a+iT]$, where a is a large negative number. In doing so, we pick up a contribution $(\log y)^{k-1}/(k-1)!$ from the residue of the integrand at $s = 0$. (To see this, expand y^s into a power series and pick out the coefficient of s^{k-1}.) The integral over the vertical component is bounded by $\leq 2T/|a|^k$, which tends to 0 as $a \to -\infty$ (with T fixed), while the integrals over the horizontal components are bounded by (using $|s|^k \geq |s|^2 = \sigma^2 + t^2$)

$$\ll \int_a^c \frac{1}{\sigma^2 + T^2} d\sigma \leq \int_a^{-T} \frac{1}{\sigma^2} d\sigma + \int_{-T}^{2} \frac{1}{T^2} dt \ll \frac{1}{T},$$

which tends to zero as $T \to \infty$. Letting first $a \to -\infty$ and then $T \to \infty$, we conclude that $I_k(y) = (\log y)^{k-1}/(k-1)!$ in the case $y \geq 1$. If $0 < y < 1$, the same argument, with a path to the right of the line $\sigma = c$, shows that $I_k(x) = 0$. This proves (1).

Disregarding questions of convergence for the moment, we obtain, using (1),

(2)

$$\begin{aligned} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\frac{x^s}{s^k} ds &= \sum_{n=1}^{\infty} f(n) \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s^k} ds \\ &= \sum_{n=1}^{\infty} f(n) I_k(x/n) = \frac{1}{(k-1)!}\sum_{n \leq x} f(n)(\log(x/n))^{k-1}. \end{aligned}$$

The integrals occurring here are absolutely convergent, and the interchaning of summation and integration is justified, provided

$$\int_{c-i\infty}^{c+i\infty} \frac{|x|^c}{|s|^k} \sum_{n=1}^{\infty} \frac{|f(n)|}{n^c} |ds| < \infty.$$

Since, by assumption, $k \geq 2$, this condition is satisfied, provided the series $F(s)$ converges absolutely at $s = c$ and $c > 0$. Thus, the above argument is valid under the condition $c > \max(0, \sigma_a)$, so (2) holds under this condition.

Problem 5

Show that if $\zeta(s)$ has no zeros in $\sigma > 1/2$ (i.e., if the Riemann Hypothesis holds) and, in addition, satisfies

$$(*) \qquad \left|\frac{1}{\zeta(s)}\right| \ll_{\epsilon,\sigma_0} |t|^{\epsilon} \quad (|t| \geq 1, \sigma \geq \sigma_0)$$

for any fixed $\epsilon > 0$ and $\sigma_0 > 1/2$, then

$$(1) \qquad \sum_{n \leq x} \mu(n) = O_{\epsilon}\left(x^{1/2+\epsilon}\right)$$

holds for any fixed $\epsilon > 0$. (Hint: Use Perron's formula for $M(\mu, x)$ (rather than the "smoothed" version of Perron's formula for $M_1(\mu, x)$). This is because the transition from M_1 to M introduces an additional error term that is larger than the square-root type error allowed in (1).)

Remarks: It can be shown (though this is beyond the scope of this class) that the Riemann Hypothesis implies $(*)$, and hence (1). In class it was shown that the converse also holds, i.e., (1) implies the Riemann Hypothesis. Thus, the Riemann Hypothesis is, in fact, equivalent to (1). In much the same way, one can show that the Riemann Hypothesis is equivalent to the assertion that the PNT holds in the form $\psi(x) = x + O_{\epsilon}(x^{1/2+\epsilon})$.

Solution

Let $\epsilon > 0$ be given, which we may assume to satisfy $0 < \epsilon < 1/4$. In the estimates below, the implied constants are allowed to depend on ϵ (but not on any other parameters or variables).

We may assume, without loss of generality, that x is of the form $x = N + 1/2$, where N is an integer. We then express $M(\mu, x)$ by Perron's formula with $a = 1 + 1/\log x$ and a suitable choice of T to be specified later:

$$(2) \qquad M(x) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT} \frac{1}{\zeta(s)}\frac{x^s}{s} ds + R(T),$$

where

$$R(T) \ll \frac{x^a}{T}\sum_{n=1}^{\infty} \frac{1}{n^a|\log(x/n)|}.$$

The integral is treated in the same way as in the proof of the PNT given in class, by shifting the path of integration, the interval $[a - iT, a + iT]$, to a path consisting of the horizontal segments $[a - iT, b - iT]$ and $[b + iT, a + iT]$ and the vertical segment $[b - iT, b + iT]$, where $b = 1/2 + \epsilon$. Note that $1/2 < b \leq 3/4$ by our assumption $0 < \epsilon \leq 1/4$.

On the vertical segment $[b - iT, b + iT]$, we have $|x^s| = x^b$. If $1 \leq |t| \leq T$, then by $(*)$, $|1/(\zeta(s)s)| \ll |t|^{\epsilon-1}$, while for $|t| \leq 1$, $|1/(\zeta(s)s)| \ll 1$ since $1/(\zeta(s)s)$ is analytic, and hence bounded, on the compact segment $[b-i, b+i]$. The integral over the vertical segment is therefore bounded by

$$\left|\int_{b-iT}^{b+iT} \frac{x^s}{\zeta(s)s} ds\right| \ll x^b \left(1 + \int_1^T t^{\epsilon-1} dt\right) \ll x^b T^\epsilon = x^{1/2+\epsilon} T^\epsilon.$$

On the horizontal segments we have $|1/\zeta(s)| \ll T^{1/2}$ (by $(*)$, applied with $\epsilon = 1/2$). Moreover, on these segments, $|x^s| \leq x^a \ll x$ and $|1/s| \leq 1/T$. Hence the integrals over the horizontal segments are bounded by

$$\left|\int_{b\pm iT, a\pm iT} \frac{1}{\zeta(s)} \frac{x^s}{s} ds\right| \ll T^{-1/2} x.$$

Altogether, we obtain the bound

$$\left|\int_{a-iT}^{a+iT} \frac{1}{\zeta(s)} \frac{x^s}{s} ds\right| \ll x^{1/2+\epsilon} T^\epsilon + \frac{x}{T^{1/2}}. \tag{3}$$

It remains to estimate the error term $R(T)$ in (2). To this end we split the sum over n into two ranges according to whether $|\log(x/n)| \geq \log 2$ or $|\log(x/n)| < \log 2$. The sum over the first range is bounded by

$$\sum_{|\log(x/n)| \geq \log 2} \frac{1}{n^a \log(x/n)} \leq (1/\log 2)\zeta(a) \ll \frac{1}{a-1} \ll \log x, \tag{4}$$

by our choice $a = 1 + 1/\log x$.

The sum over the second (finite) range extends over n with $x/2 < n < 2x$. To deal with this range, we set $n_0 = [x]$ and $n = n_0 + h$. Note that, by our assumption that x is a half-integer, we have $x = n_0 + 1/2$. The range $x/2 < n < 2x$ is then equivalent to $n_0/2 < n \leq 2n_0$, or $-n_0/2 \leq h \leq n_0$. In the latter range we have

$$\begin{aligned} |\log(x/n)| = |\log(n/x)| &= \left|\log \frac{n_0 + h}{n_0 + 1/2}\right| = \left|\log\left(1 + \frac{h - 1/2}{n_0 + 1/2}\right)\right| \\ &\gg \frac{|h - 1/2|}{n_0 + 1/2} = \frac{|h - 1/2|}{x} \end{aligned}$$

since for $-1/2 \leq t \leq 2$, $|\log(1+t)| \asymp |t|$. Hence,

$$\begin{aligned} \sum_{|\log(x/n)| < \log 2} \frac{1}{n^a |\log(x/n)|} &\ll \frac{1}{x^a} \sum_{-n_0/2 < h \leq n_0} \frac{1}{|h - 1/2|/x} \\ &\ll x^{1-a} \sum_{|h| \leq x} \frac{1}{|2h - 1|} \ll x^{1-a} \log(2x) \ll \log x. \end{aligned} \tag{5}$$

From (4) and (5) we get

$$R(T) \ll \frac{x^a}{T} \log x \ll \frac{x}{T} \log x. \tag{6}$$

Choosing now $T = x$, the bounds (3) and (6) both become $\ll x^{1/2+3\epsilon}$, which is the desired bound, after replacing ϵ by $\epsilon/3$.