

Math 531, Fall 2005  
Analytic Number Theory  
Problem Set 7  
Due December 5, 2005

**Problem 1**

Show that if  $f$  is a periodic, completely multiplicative arithmetic function, then  $f$  is a Dirichlet character to some modulus  $q$ .

**Problem 2**

Show that if every arithmetic progression  $a \pmod q$  with  $(a, q) = 1$  contains at least one prime, then every such progression contains *infinitely* many primes.

### Problem 3

Given a rational number  $a$  with  $0 < a \leq 1$ , define  $\zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s}$ . Show that any Dirichlet  $L$ -function can be expressed in terms of the functions  $\zeta(s, a)$ , and that, conversely, any such function  $\zeta(s, a)$  with rational  $a$  can be expressed in terms of Dirichlet  $L$ -functions.

**Problem 4\***

Define the Dirichlet resp. logarithmic densities of a set  $A \subset \mathbb{N}$  by the formulas

$$D(A) = \lim_{\sigma \rightarrow 1^+} (\sigma - 1) \sum_{n \in A} \frac{1}{n^\sigma}$$

and

$$\delta(A) = \lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{n \leq x, n \in A} \frac{1}{n},$$

if the limits exist. Show that if  $\delta(A)$  exists then  $D(A)$  exists.

### Problem 5

(Bonus problem) Last, but not least, here is one that seems hopelessly difficult at first sight (and which won't on the final and is quite unlikely to show up on comp exams), but ...

One of the most famous unsolved problems in number theory is Goldbach's conjecture according to which every even integer greater than 2 can be expressed as the sum of two primes. The conjecture has been numerically verified for all even numbers up to about  $10^{14}$ , but a proof that the conjecture holds for all even numbers remains as elusive as ever. Prove that there exists a bound  $C$ , such that if the conjecture holds for even numbers up to  $C$ , then it holds for all even numbers.