Asymptotic Analysis
Problem Set 1
Due February 8, 2005

Remarks. The homework problems vary in nature and difficulty. Some are routine (and maybe boring), while others are more challenging (and perhaps also a bit more interesting); the latter problems are indicated by an asterisk. While most problems are related to the class material, some are independent of this material, and their solution requires some degree of ingenuity and independent thinking rather than applying particular techniques or theorems.

The write-up is an important part of your solution and will be taken into account in the grading. Try to write up your solutions in a "professional" manner, with complete sentences and proper punctuation. Your write-up should contain all necessary steps in a logical order, without becoming excessively verbose.

Problem 1*

Let n be an integer ≥ 2 and p a positive real number. In class it was shown (at least in the case $n = 2$) that

$$\left(\sum_{i=1}^{n} a_i\right)^p \asymp_{n,p} \sum_{i=1}^{n} a_i^p \quad (a_1, a_2, \ldots, a_n > 0).$$

Let $c_1(n,p)$ and $c_2(n,p)$ denote the best-possible lower and upper bound constants in this estimate. Determine these "optimal" constants.

Problem 2

Prove the following estimates rigorously, with explicit values for the constants.

(i) $\log x \ll 1/x$ *(*$0 < x < 1$*).*

(ii) $\log(1+x) = x + O_c(x^2)$ *(*$|x| \le c$*,* c *any fixed constant with* $0 < c < 1$*).*

Problem 3

Prove that

$$\left(x + 1 + O\left(\frac{1}{x}\right)\right)^x = ex^x\left(1 + O\left(\frac{1}{x}\right)\right)$$

(Begin by stating, in a precise manner, the correct interpretation of the above relation.)

Problem 4

Let

$$I(x,\alpha) = \int_1^x \frac{\sin(\alpha t)}{t}\,dt,$$

where α *is a fixed real and non-zero number. Use integration by parts to show that* $I(x,\alpha)$ *converges as* $x \to \infty$*, with limit* $I(\alpha)$*, say, and show that* $I(x,\alpha) = I(\alpha) + O_\alpha(1/x)$*.*

Problem 5

Let

$$I(x) = \int_1^x \left(1 + \frac{1}{t}\right)^t dt.$$

Obtain an asymptotic estimate for $I(x)$ *with error* $O(1)$*.*

Problem 6*

Show that if $f(x)$ *satisfies* $f(x) = x^2 + O(x)$*, and* f *is differentiable* with nondecreasing derivative $f'(x)$ *for sufficiently large* x*, then* $f'(x) = 2x + O(\sqrt{x})$*.*

Remark. While O-estimates can be integrated provided the range of integration is contained in the range of validity of the estimate, in general such estimates cannot be differentiated. The above problem illustrates a situation where, under certain additional conditions (namely, the monoticity of the derivative), differentiation of a O-estimate is allowed.

Problem 7

Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with nonnegative terms a_n.

(i) Show that there exists a real-valued function $\psi(n)$ with $\lim_{n\to\infty} \psi(n) = \infty$ such that the series $\sum_{n=1}^{\infty} a_n \psi(n)$ still converges.

(ii) Is the result true without the assumption that the terms a_n be nonnegative (i.e., assuming only that the series $\sum_{n=1}^{\infty} a_n$ converges)?*