Asymptotic Analysis
Problem Set 2
Due March 1, 2005

Problem 8

[DeBruijn 2.1] Show that the equation $\sin x = (\log x)^{-1}$ *has just one root* x_n *in the interval* $2\pi n < x_n < 2\pi n + \pi/2$ *(*$n = 1, 2, 3, \ldots$*), and that*

$$x_n = 2\pi n + (\log 2\pi n)^{-1} + O((\log 2\pi n)^{-3}) \quad (n \to \infty).$$

Problem 9

[DeBruijn 2.2] Let $f(t)$ *be positive, and assume that*

$$e^{tf(t)} = f(t) + t + O(1) \quad (0 < t < \infty).$$

Show that

$$f(t) = t^{-1} \log t + O(t^{-2}) \quad (t \to \infty).$$

Problem 10

For $u > 1$*, let* $\xi = \xi(u)$ *denote the unique positive solution to the equation* $\int_0^1 e^{\xi t} dt = u$*. Obtain an asymptotic estimate for* $\xi(u)$*, as* $u \to \infty$*, with an error term* $O((\log \log u)/(\log u)^2)$*.*

Problem 11

Let $x_0 = 1$ *and for* $n \geq 1$ *let* $x_n = \sin x_{n-1}$*. (This is the sequence obtained by repeatedly pressing the "sin" button on a calculator, starting with initial value* 1*.) It is easy to see that* x_n *converges to* 0 *as* $n \to \infty$*. Obtain an estimate for* x_n *consisting of a main term and an error term of smaller order (such as* $x_n = (1/n)(1 + O(1/n))$*).*

Problem 12

Show that if $f(x)$ *satisfies*

$$f(x) = O_\epsilon(x^\epsilon) \quad (x \geq x_0(\epsilon)),$$

for any fixed positive number ϵ*, with a suitable constant* $x_0(\epsilon)$*, then there exists a function* $\epsilon(x) > 0$ *with* $\lim_{x\to\infty} \epsilon(x) = 0$*, such that* $f(x)$ *satisfies the* single *estimate*

$$f(x) = O(x^{\epsilon(x)}) \quad (x \geq x_0),$$

with a suitable x_0*.*

Problem 13

Let $\sum_{n=1}^{\infty} a_n$ *be a series with real terms* a_n*, and suppose that* $a_n \to 0$ *as* $n \to \infty$*. Let* $S_N = \sum_{n=1}^{N} a_n$ *denote the partial sums of this series, and let* $\underline{S} = \liminf_{N\to\infty} S_N$ *and* $\overline{S} = \limsup_{N\to\infty} S_N$ *(where* $\overline{S}$ *and* $\underline{S}$ *may be infinite). Prove that, for every* $x \in (\underline{S}, \overline{S})$*, there exists a sequence* $N_1 < N_2 < \dots$ *such that* $\lim_{k\to\infty} S_{N_k} = x$*.*