Asymptotic Analysis
Problem Set 3
Due March 15, 2005

Problem 14

Let $S_N = \sum_{n=1}^{N} n^n$. *Obtain an asymptotic estimate for* S_N *with relative error* $O(1/N^2)$.

Problem 15

Given an arithmetic function $a(n)$, $n = 1, 2, \dots,$ *and a real number* $\alpha > -1$ *define a mean value* $M_\alpha(a)$ *by*

$$M_\alpha(a) = \lim_{x\to\infty} \frac{1+\alpha}{x^{1+\alpha}} \sum_{n\le x} n^\alpha a(n),$$

provided the limit exist. (In particular, $M_0(a) = M(a)$ *is the usual asymptotic mean value of a.) Prove, using a rigorous* $\epsilon - x_0$ *argument, that the mean value* $M_\alpha(a)$ *exists if and only if the ordinary mean value* $M(a) = M_0(a)$ *exists. (As a consequence, if one of the mean values* $M_\alpha(a)$, $\alpha > -1$, *exists, then all of these mean values exist.)*

Problem 16

[De Bruijn 3.4] Let

$$s_n = \frac{1}{\log n} - \frac{1}{\log(n+1)} + \frac{1}{\log(n+2)} - \cdots .$$

Show that

$$s_n = \frac{1}{2\log n} + O\left(\frac{1}{n\log^2 n}\right) \quad (n\to\infty).$$

Problem 17

Given a real number $p > 0$, *let*

$$S_p(x) = \sum_{n=0}^{\infty} x^{n^p}.$$

Obtain an asymptotic estimate, with error term, for $S_p(x)$ *as* $x \to 1-$.

Problem 18

[De Bruijn 3.1] Show that

$$\sum_{n=1}^{\infty} \frac{e^{-n^2 t}}{n} = -\frac{1}{2}\log t + \frac{1}{2}\gamma + O(\sqrt{t}) \quad (0 < t < 1),$$

where $\gamma = -\int_0^\infty e^{-x} \log x dx$ *is Euler's constant.*

Problem 19*

Using the integral test, one can easily show that each of the series

$$S_0 = \sum \frac{1}{n}, \quad S_1 = \sum \frac{1}{n \log n}, \quad S_2 = \sum \frac{1}{n \log n \log\log n}, \ldots$$

where the summation runs over all sufficiently large n *(large enough so that the iterated logarithms are defined and bounded away from* 0*), diverges. Can one obtain a convergent series by letting the number of log factors in the denominator tend to infinity with* n*? More precisely, let* $\log_k x$ *denote the* k*-fold iterated logarithm, and for* $k \geq e$ *let* $k(x)$ *be the largest integer* k *such that* $\log_k x \geq 1$. *Then the series*

$$S = \sum_{n=3}^{\infty} \frac{1}{n \log n \ldots \log_{k(n)} n},$$

is well-defined. Determine whether this series converges or diverges.