

UIUC Department of Mathematics

PROBLEM OF THE WEEK

October 14, 2007

Record sequences

Here is a problem inspired by the record-setting temperatures we recently experienced. Consider a particular day of the year, say, October 31, and suppose we have 100 years of recorded temperatures for this day, from 1907 till 2006. Let t_1, t_2, \dots, t_{100} denote the sequence of temperature readings for October 31 recorded during this period. Thus, t_1 is the temperature recorded on Halloween of 1907, t_2 the temperature recorded on Halloween of 1908, and so on. Assume that the numbers t_n are all pairwise distinct, and call an element t_n in this sequence a “record” if it is either greater than all preceding values (a “record high”) or less than all preceding values (a “record low”).

The first element, t_1 , is, by default, a record. Also, the second element, t_2 , obviously always represents a new record—a record high if it is greater than t_1 , and a record low if it is less than t_1 . However, beyond the second element the situation gets murkier, though we certainly expect records to become less and less likely the further down the sequence we get. How (un)likely is it that *every* term in the sequence is a record value? More precisely:

What is the probability that, given a sequence of 100 pairwise distinct real numbers, every single element in this sequence represents a record in the above sense?

—Turn Page for Solution—

Solution to “Record sequences”

We will show that, given a sequence of n distinct real numbers, the probability that this sequence is a record sequence is $2^{n-1}/n!$. For the value $n = 100$ of the problem, this gives $2^{99}/100! \approx 6.8 \cdot 10^{-129}$, an astronomically small value.

To prove this claim, we first make precise what we mean by “probability”. The natural interpretation is as the probability that a random permutation of n distinct numbers is a record sequence in the sense of the problem, i.e., the number of permutations that are record sequences divided by the total number of permutations. Since altogether there are $n!$ permutations, what we need to show is the following:

(*) *Given n distinct real numbers, there are exactly 2^{n-1} permutations of these numbers that produce a record sequence.*

If one tries to count record permutations by counting the number of choices for the first position, the number of choices for the second position, etc., one quickly runs into insurmountable problems. However, the counting becomes nearly trivial if we proceed backwards, i.e., count the number of choices for the last position, that for the second-last position, etc. Since the last element has to be a record, it must be either the largest or the smallest among the n given numbers. Thus, there are exactly 2 ways to fill the last position. Once this position is filled, the second-last element must be either the smallest or the largest among the remaining $n - 1$ numbers, so there are again exactly 2 ways to fill this position. Continuing in this manner, we see that for each of the positions $n, n - 1, \dots, 2$ there are exactly 2 choices. Once these $n - 1$ positions have been filled, there is a single choice left to fill the first position. Thus, altogether, there are 2^{n-1} ways to arrange the given numbers in a record sequence. This proves (*).

PROBLEM OF THE WEEK ARCHIVE

<http://www.math.uiuc.edu/~hildebr/pow>