University of Illinois
Department of Mathematics

PROBLEM OF THE WEEK
October 12, 2005

A tower of x's

Determine

$$\lim_{x \to 0+} x^{*2005},$$

where

$$x^{*n} = \underbrace{x^{x^{\cdots^{x}}}}_{n}$$

denotes a "tower" of n x's, with the order of the exponentiations from top to bottom (so that, for example, $x^{*3} = x^{(x^x)}$).

—*Turn Page for Solution*—

Solution to "A tower of x's"

We will show that the limit in question is 0. More generally, we will show that the limit $L_n = \lim_{x\to 0+} x^{*n}$ exists for all positive integers n, and is equal to 0 if n is odd and 1 if n is even.

The first two cases are easy: If $n=1$, then $x^{*n}=x$, so $L_1=0$. For $n=2$, $x^{*2}=x^x$, so $\log x^{*2} = \log x^x = x\log x$, which converges to 0 by l'Hopital's Rule. Hence $L_2 = e^0 = 1$.

Next, note that if $L_n = 1$, then $x^{*n} \ge 1/2$ for sufficiently small x, and so $x^{*(n+1)} = x^{x^{*n}} \le x^{1/2}$ for such x, and hence $L_{n+1} = \lim_{x\to 0+} x^{*(n+1)} = 0$. Thus, to prove the above assertion, it remains to show that $L_n = 1$ for even values of n.

We prove this by induction. As noted above, the result holds for $n=2$, so let $n\ge 2$ be an even integer such that $L_n = 1$. We need to show that $L_{n+2}=1$, or equivalently, $(*)$ $\lim_{x\to 0+} \log x^{*(n+2)} = 0$.

We have, for $0<x<1$,

$$|\log x^{*(n+2)}| = x^{*(n+1)}|\log x| = x^{x^{*n}}|\log x|.$$

If x is sufficiently small, then by the induction hypothesis $x^{*n}\ge 1/2$ and hence $x^{x^{*n}} \le x^{1/2}$. Since

$$\lim_{x\to 0} x^{1/2}\log x = 2\lim_{x\to 0} x^{1/2}\log x^{1/2} = 0,$$

we obtain $(*)$ as claimed.