University of Illinois
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PROBLEM OF THE WEEK
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Fibonacci numbers ending in 9's

Show that, for every k, there exists a Fibonacci number whose decimal representation ends in k 9's. (The Fibonacci numbers are defined recursively by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$.

—Turn Page for Solution—

Solution to "Fibonacci numbers ending in 9's"

First note that an integer ends in a string of k 9's in its decimal representation if and only if it is congruent to $\underbrace{99\dots9}_{k}$ modulo 10^k, or equivalently congruent to -1 modulo 10^k. Thus, the problem amounts to showing that, for every k, there exists a Fibonacci number congruent to -1 modulo 10^k.

Next, observe that, by setting $F_n = F_{n+2} - F_{n+1}$, we can continue the Fibonacci sequence "backwards" and define F_n for all integers n, including negative values of n. By construction, this extended sequence satisfies the Fibonacci recursion $F_{n+2} = F_n + F_{n+1}$ for all integers n.

The first few terms with negative indices are easy to calculate: $F_{-1} = 1 - 1 = 0$, $F_{-2} = 1 - 0 = 1$, $F_{-3} = 0 - 1 = -1$, $F_{-4} = 1 - (-1) = 2$, etc. In particular, the term $F_{-3} = -1$ is (trivially) congruent to -1 modulo 10^k for all positive integers k, so the extended Fibonacci sequence contains at least one term with the desired congruence property.

To prove the existence of *ordinary* Fibonacci numbers congruent to -1 modulo 10^k, for any given value of k, we will show that, for any modulus m, the sequence $\{F_n \bmod m\}_{n=-\infty}^{\infty}$, is periodic. Since, for each k, we know that there is at least one term F_n that is congruent to -1 modulo 10^k, this implies that there are infinitely many such terms with positive index n satisfying the same congruence.

To prove the asserted periodicity of $F_n \bmod m$, let a modulus m be given, and let $f_n \in \{0, 1, \dots, m-1\}$ denote the remainder of F_n modulo m. Since there are only finitely many (namely, at most m^2) possible values for the tuples (f_n, f_{n+1}), by the pigeonhole principle two of these tuples must be equal, i.e., there exist integers n and $p \geq 1$ such that $f_n = f_{n+p}$ and $f_{n+1} = f_{n+p+1}$. The Fibonacci recursion then implies $f_{n+i} = f_{n+p+i}$ for all $i \geq 0$, and using the "backwards recursion" $F_n = F_{n+2} - F_{n+1}$, we see that the same holds for all $i < 0$. But this means that $\{f_n\}$ is periodic, with period p, as claimed.