UIUC Department of Mathematics

PROBLEM OF THE WEEK
November 29, 2005

Euclidean primes

Euclid's famous proof of the infinitude of prime numbers hinges on the fact that if $p_1, \ldots, p_n$ are primes, then the number $p_1 \ldots p_n + 1$ must be either a prime or divisible by a prime other than $p_1, \ldots, p_n$. This suggests to define a sequence of "Euclidean primes" a_n by letting $a_1 = 2$ and, for $n \geq 1$, $a_{n+1} = a_1 \ldots a_n + 1$. The first few terms of this sequence are $2, 3, 7, 43, 1807, 3263443, \ldots$.

Show that

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = 1.$$

—*Turn Page for Solution*—

Solution to "Euclidean primes"

The key is to express the sum in question as a telescoping sum. Let a_n denote the n-th term in the given sequence, and let P_n be the product of the first n terms. Then $a_{n+1} = 1 + P_n$ and $P_{n+1} = a_{n+1}P_n$, from which it easily follows that $1/a_{n+1} = 1/P_n - 1/P_{n+1}$ for $n = 1, 2, \ldots$. Hence the sum over $1/a_{n+1}$, $n = 1, 2, \ldots$, telescopes and is equal to $1/P_1 = 1/2$. Adding in the term $1/a_1 = 1$ shows that the sum over all terms $1/a_n$ is equal to 1.