UIUC Department of Mathematics

PROBLEM OF THE WEEK
October 8, 2006

The Fibonacci constant

Start with the famous Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, \dots$$

(defined by $F(1) = 1$, $F(2) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 3$). Drop all but the last digit in each of these numbers, and interpret the resulting sequence of digits as the decimal expansion of a certain real number $F \in (0, 1)$, which we may term the "Fibonacci constant":

$$F = 0.112358314594370774\dots$$

In other words, F is the real number in the interval $(0, 1)$ whose nth digit after the decimal point is the last digit of the n-th Fibonacci number $F(n)$. The question now is simple:

Is F rational?

—Turn Page for Solution—

Solution to "The Fibonacci constant"

We will show that F is rational. This is equivalent to showing that the decimal expansion of F is eventually periodic.

By definition, $F = 0.f_1 f_2 f_3 \dots$, where f_n is the last digit of the n-th Fibonacci number $F(n)$, i.e., f_n *is the residue of* $F(n)$ *modulu* 10. In particular, we have $f_n \equiv F(n)$ modulo 10, and hence, by the recurrence for $F(n)$,

$$(*) \qquad f_{n+2} \equiv f_{n+1} + f_n \bmod 10 \quad (n = 1, 2, \dots).$$

Since f_{n+2} must be an element of $\{0, 1, \dots, 9\}$, the congruence $(*)$ determines f_{n+2} uniquely, for any given pair of values (f_n, f_{n+1}). By induction it follows that any given pair of consecutive values (f_n, f_{n+1}) determines the entire remainder of the sequence, i.e., f_{n+i} for all $i = 1, 2, \dots$.

Now note that each pair (f_n, f_{n+1}) must be of the form (a, b) with both $a, b \in \{0, 1, \dots, 9\}$. Since there only finitely many (namely, 10^2) such pairs (a, b), by the pigeonhole principle there exist indices $m < n$ with $(f_m, f_{m+1}) = (f_n, f_{n+1})$. Setting $p = n - m$, we then have $f_{m+p+i} = f_{m+i}$ for $i = 0, 1$, and by the above observation, this relation persists for all $i = 0, 1, \dots$. Thus the sequence $\{f_i\}$ is eventually periodic with period p, as we wanted to show.