

# 2011 U OF I FRESHMAN MATH CONTEST

1. Let  $x = 0.12345678910111213\dots$  be the number whose decimal expansion consists of the sequence of natural numbers written next to each other.

(a) Determine the 2011th digit after the decimal point of  $x$ .

(b) Prove that  $x$  is irrational.

2. Find, with proof, a simple formula for the sum

$$S_n = \sum_{k=1}^n \frac{k}{(k+1)!}.$$

3. There are 92 airports in Illinois. Suppose that from each of these airports a plane takes off and flies to the nearest neighboring airport. Assuming the mutual distances between the airports are all distinct prove that there is no airport at which more than five planes land.

4. Find, with proof, a simple formula for the sum

$$\sum_{k=0}^n \binom{n+k}{k} 2^{-k}.$$

5. [A2, Putnam 1980] Given positive integers  $r$  and  $s$ , let  $f(r, s)$  denote the number of 4-tuples of positive integers  $(a, b, c, d)$  such that the least common multiple of any **three** of these integers is equal to  $3^r 7^s$  (i.e., such that  $[a, b, c] = [a, b, d] = [a, c, d] = [b, c, d] = 3^r 7^s$ , where  $[\dots]$  denotes the least common multiple). Find, with proof, a simple formula for  $f(r, s)$ .

6. Find, with proof, the sum of the infinite series

$$\frac{4}{4^2 - 1} + \frac{4^2}{4^4 - 1} + \frac{4^4}{4^8 - 1} + \frac{4^8}{4^{16} - 1} + \dots$$

[Solutions at <http://www.math.uiuc.edu/contests.html>]