UIUC Mock Putnam Exam 1/2002
Solutions

Problem 1. Show that $\log_{2002} 2003 + \log_{2003} 2002 > 2$.

Solution. Setting $a = \log 2002$ and $b = \log 2003$, where log denotes the natural logarithm, the inequality to be proved becomes $(a/b) + (b/a) > 2$. Since a and b are positive, this is equivalent to $a^2 + b^2 > 2ab$, and the latter inequality holds since $0 < (a - b)^2 = a^2 + b^2 - 2ab$.

Problem 2. Show that the number

$$\frac{2^3}{3^2} + \frac{4^3}{5^2} + \frac{6^3}{7^2} + \cdots + \frac{2002^3}{2003^2}$$

is not an integer.

Solution. The key observation here is that 2003 is a prime number. If we write the sum as a single fraction, the common denominator will involve a factor 2003, whereas the numerator is a sum of terms all of which, *except for the last one*, are divisible by 2003. Hence, after cancelling any common (prime) factors, there remains a factor 2003 in the denominator. Thus the given sum cannot be an integer.

Problem 3. Find the sum of the infinite series

$$\sum_{n=3}^{\infty} \frac{1}{\binom{n}{3}}.$$

Solution. The sum is equal to 3/2. To see this, use the identity

$$\frac{2}{n(n-1)(n-2)} = \frac{1}{(n-1)(n-2)} - \frac{1}{n(n-1)}$$

to convert the sum into one that "telescopes":

$$\sum_{n=3}^{\infty} \frac{1}{\binom{n}{3}} = \sum_{n=3}^{\infty} \frac{6}{n(n-1)(n-2)} = \sum_{n=3}^{\infty} \left(\frac{3}{(n-1)(n-2)} - \frac{3}{n(n-1)} \right)$$
$$= \sum_{m=2}^{\infty} \frac{3}{m(m-1)} - \sum_{n=3}^{\infty} \frac{3}{n(n-1)} = \frac{3}{2}.$$

(Note here that splitting the infinite series into two series is permissible since the individual series are both absolutely convergent. However, this would no longer be the case if one would go one step further and write, for example, $1/n(n-1) = 1/(n-1) - 1/n$, since then the resulting invidual series would be divergent.)

Problem 4. Find all positive integers n which are equal to the sum of digits of n^2.

Solution. We will show that $n = 1$ and $n = 9$ are the only such integers. To see this, suppose that n has k digits. Then, on the one hand $n \geq 10^{k-1}$, and on the other hand $n^2 < 10^{2k}$, so the sum of digits of n^2 is at most $2 \cdot 9k = 18k$. Thus if n is a solution to the problem, then $10^{k-1} \leq 18k$, which implies $k \leq 2$. (One way to see this is by noting that the function $f(x) = 10^{x-1}/(18x)$ is > 1 at $x = 3$ and is increasing for $x > 3$.) Therefore $n < 10^2$, so $n^2 < 10^4$, and hence the sum of digits of n^2 is at most $4 \cdot 9 = 36$. Thus any solution n is at most 36 (in fact, strictly less than 36 since 9999 is not a square). To further reduce the number of cases to be checked, we use the fact that the sum of digits of a number is congruent to this number modulo 9. Thus, any solution n to the problem must satisfy $n \equiv n^2 \pmod 9$, and this is only possible if $n \equiv 1 \pmod 9$ or $n \equiv 0 \pmod 9$. Thus, among the integers less than 36 only $n = 1, 9, 10, 18, 19, 27, 28$ remain as possible solutions. Checking these numbers directly then easily shows that only $n = 1$ and $n = 9$ are solutions.

Problem 5. Is there a prime number p and a sequence $(P_n)_{n \in \mathbf{N}}$ of integer points (i.e., points with integer coordinates) in the plane such that every integer point in the plane belongs to the sequence $(P_n)_{n \in \mathbf{N}}$ and the distance between any two consecutive points P_n and P_{n+1} equals p ?

Solution. The answer is "yes"; for example, we can take $p = 5$. To prove this, first observe that it is enough to show that we can get from the origin to the point $(1, 0)$. For suppose we have a path of the desired type from $(0, 0)$ to $(1, 0)$. By rotating this path by 90, 180, or 270 degrees, we can then also reach the points $(0, 1)$, $(-1, 0)$, and $(0, -1)$ from the origin. By repeated applications of such moves, we reach any point with integer coordinates.

Now, the path $(0, 0) \to (-5, 0) \to (-2, 4) \to (1, 0)$ is a path of the desired form, i.e., it leads from the origin to the point $(1, 0)$ via integer points, in moves of length 5.

Further remarks. Clearly $p = 2$ will not work, because then the only possible movements are horizontal or vertical steps of length 2, and so if P_0 is the origin, say, then the only possible coordinates of each P_n will be even integers. Similarly for $p = 3$ only horizontal or vertical moves in steps of 3 are possible, so starting at the origin we can only reach points whose coordinates are multiples of 3. The key property of the prime $p = 5$ is that it is equal to the *diagonal* distance between two integer points, namely $5 = \sqrt{3^2 + 4^2}$. This enables diagonal moves of length 5. More generally, it is known from number theory that any prime $p \equiv 1 \pmod 4$ has this property.