UIUC Mock Putnam Exam 2/2002

Problem 1. Determine all prime numbers in the sequence $101, 10101, 1010101, \ldots$.

Problem 2. Let b be a positive real number such that

$$1 + 2b + 3b^2 + \cdots + nb^{n-1} + \cdots = 2002.$$

Which number is larger: $4004b$ or $2002b^2 + 2001$?

Problem 3. Find a polynomial $f(x)$ with real coefficients, of degree ≤ 2, which best approximates $\sin x$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, in the sense that the integral

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (f(x) - \sin x)^2 \, dx$$

is as small as possible.

Problem 4. Let $a_1, a_2, \ldots, a_{2n+1}$ be integers with the property that if we remove any one of these numbers, we can divide the remaining $2n$ numbers into two groups of n numbers each, having the same sum. Show that $a_1 = a_2 = \cdots = a_{2n+1}$.