UIUC Mock Putnam Exam 2/2002

Problem 1. Determine all prime numbers in the sequence $101, 10101, 1010101, \ldots$.

Solution. The first element, 101, is obviously a prime. We now show that all other elements in this sequence are composite. The key observation is that the nth term in this sequence is given by a finite geometric series, namely

$$a_n = \sum_{k=0}^{n} 100^k = \frac{100^{n+1} - 1}{100 - 1}$$

The numerator in this fraction can be factored into $(10^{n+1} - 1)(10^{n+1} + 1)$, whereas the denominator is equal to $99 = 3^2 \cdot 11$. Since a_n is an integer, all prime factors in the denominator must appear in one or both of the factors in the numerator. However, since for $n \geq 2$ the numbers $10^{n+1} \pm 1$ are both greater than the denominator, after cancelling out any common factors, we are left with a factorization of a_n into two integer factors, each greater than 2. Hence, a_n is composite for all $n \geq 2$.

Problem 2. Let b be a positive real number such that

$$1 + 2b + 3b^2 + \cdots + nb^{n-1} + \cdots = 2002.$$

Which number is larger: $4004b$ or $2002b^2 + 2001$?

Solution. The answer is "neither": both expresssions are equal. To see this, start with the formula for the sum of a geometric series

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

which is valid when $|x| < 1$. Either by squaring or by taking derivatives, it follows that

$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}$$

for $|x| < 1$. Thus b satisfies $1/(1-b)^2 = 2002$. Clearing denominators, we obtain

$$1 = 2002(1-b)^2 = 2002 - 4004b + 2002b^2,$$

so $4004b = 2002b^2 + 2001$.

Problem 3. Find a polynomial $f(x)$ with real coefficients, of degree ≤ 2, which best approximates $\sin x$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, in the sense that the integral

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (f(x) - \sin x)^2 \, dx$$

is as small as possible.

Solution. Letting $f(x) = ax^2 + bx + c$, we have

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} \left(ax^2 + bx + c - \sin x\right)^2 dx \\ &= \int_{-\pi/2}^{\pi/2} \left((ax^2 + c)^2 + (bx - \sin x)^2 + 2(ax^2 + c)(bx - \sin x)\right) dx. \end{aligned}$$

Since $(ax^2+c)(bx-\sin x)$ is an odd function of x, the integral over this term vanishes, so I reduces to

$$I = \int_{-\pi/2}^{\pi/2} (ax^2 + c)^2 dx + \int_{-\pi/2}^{\pi/2} (bx - \sin x)^2 dx.$$

Clearly the optimal choice for a and c is $a = c = 0$. Thus, I reduces to

$$I = \int_{-\pi/2}^{\pi/2} (bx - \sin x)^2 dx.$$

Using the evaluations

$$\begin{aligned} &\int_{-\pi/2}^{\pi/2} x^2 dx = \frac{\pi^3}{12}, \quad \int_{-\pi/2}^{\pi/2} \sin^2 x dx = \frac{\pi}{2}, \\ &\int_{-\pi/2}^{\pi/2} x \sin x dx = -x \cos x \Big|_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \cos x dx = 2, \end{aligned}$$

we obtain

$$I = \frac{\pi^3}{12} b^2 - 4b + \frac{\pi}{2}.$$

Differentiating with respect to b and setting the derivative equal to zero, we see that the latter expression is minimal when $b = 24/\pi^3$. The desired polynomial is therefore $f(x) = (24/\pi^3)x$.

Problem 4. Let $a_1, a_2, \ldots, a_{2n+1}$ be integers with the property that if we remove any one of these numbers, we can divide the remaining $2n$ numbers into two groups of n numbers each, having the same sum. Show that $a_1 = a_2 = \cdots = a_{2n+1}$.

Solution. Clearly, for any integer a, the numbers $a_1+a, a_2+a, \ldots, a_{2n+1}+a$ also satisfy the conditions of the problem. We may therefore assume that one of the given numbers is zero, and we need to show that all of them are zero.

Let S denote the sum of all $2n+1$ numbers a_i. The hypothesis implies that, for each i, $S - a_i$ is twice an integer, and hence must be even. This is only possible if all a_i have the same parity. Since, by our assumption, one of the numbers a_i is zero, all numbers a_i must be even. Dividing each a_i by 2, we obtain a new set of numbers $a_i' = a_i/2$, $i = 1, \dots, 2n+1$, which again satisfies the conditions of the problem, with at least one of the numbers being zero. We conclude as before that the numbers a_i' must all be even, i.e., that 2 divides a_i' for all i, or equivalently, that 2^2 divides a_i for all i. Continuing this process, we see that, for any positive integer k, 2^k divides all a_i. But this is only possible if all a_i are zero.