UIUC Mock Putnam Exam 3/2002

Problem 1. Evaluate the infinite series

$$S = \sum_{n=0}^{\infty} \frac{n+1}{n!}.$$

Problem 2. Suppose every point in the plane is colored with one of three colors. Show that, for any positive real number d, there exist two points of the same color and of mutual distance d.

Problem 3. Evaluate the sum

$$\sum_{n=1}^{1003002} \frac{1}{\langle \sqrt{n} \rangle},$$

where $\langle x \rangle$ denotes the integer closest to x.

Problem 4. Let S be a set of prime numbers which contains the number 2003 and has the property that for any distinct elements $q_1, q_2, \ldots, q_n$ of S, any prime factor of $q_1 q_2 \cdots q_n - 1$ belongs to S. Show that S consists of the entire set of prime numbers.

Problem 5. Let $a_n = \lfloor (1+\sqrt{2})^n \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Prove that a_n is odd if n is even, and even if n is odd.