UIUC Mock Putnam Exam 3/2002
Solutions

Problem 1. Evaluate the infinite series

$$S = \sum_{n=0}^{\infty} \frac{n+1}{n!}.$$

Solution. Multiplying the exponential series $e^x = \sum_{n=0}^{\infty} x^n/n!$ by x and differentiating, we get $\sum_{n=0}^{\infty}(n+1)x^n/n! = (xe^x)' = e^x(1+x)$. Setting $x = 1$ shows that $S = 2e$. [An alternative approach would be to write the terms in the given series as $(n+1)/n! = 1/(n-1)! + 1/n!$ (for $n \geq 1$) and split up the resulting series. This leads to $S = 2\sum_{n=0}^{\infty} 1/n!$, which again is equal to $2e$ by the exponential series.]

Problem 2. Suppose every point in the plane is colored with one of three colors. Show that, for any positive real number d, there exist two points of the same color and of mutual distance d.

Solution. We argue by contradiction. Assume that there exists $d > 0$ such that no two points of mutual distance d have the same color. Then the vertices of any equilateral triangle of side d must have distinct colors. Moreover, by considering a rhombus formed of two such triangles sharing one side, say ABC and BCD, we see that A and D must have the same color (since the colors at A and D must be different from those at both B and C, and there are only three colors available). Now, elementary trigonometry shows that the two points A and D are $\sqrt{3}d$ apart. Since the above argument applies to any rhombus formed of two equilateral triangles with side d, it follows that *any* two points of distance $\sqrt{3}d$ must have the same color. In particular, if we consider a circle of radius $\sqrt{3}d$, it follows that all points on this circle must have the same color as its center. But since the diameter of the circle is greater than d, there clearly exist two points on the circle that are a distance d apart from each other. Since these two points have the same color, we have reached the desired contradiction.

Problem 3. Evaluate the sum

$$\sum_{n=1}^{1003002} \frac{1}{\langle\sqrt{n}\rangle},$$

where $\langle x\rangle$ denotes the integer closest to x.

Solution. We show that the sum equals 2002. More generally, if $S(n) = \sum_{k=1}^{n} 1/\langle\sqrt{k}\rangle$, we show that, for any positive integer n, $(*)$ $S(n(n+1)) = 2n$. Since $1003002 = 1001 \cdot 1002$, this gives the result by taking $n = 1001$.

To prove $(*)$, note first that $\langle\sqrt{k}\rangle$ is equal to a positive integer m if and only if $\sqrt{k}$ is in the interval $(m-1/2, m+1/2)$, or equivalently, if and only if k falls in the interval $I_m = (m^2 - m + 1/4, m^2 + m + 1/4)$. Now I_m contains exactly $2m$ integers, namely $m^2 - m + 1, \ldots, m^2 + m$. Furthermore, since $(m+1)^2 - (m+1) + 1 = m^2 + m + 1$, the intervals I_m, $m = 1, 2, \ldots, n$, cover adjacent ranges of integers. Since the smallest integer in I_1 is $1^2 - 1 + 1 = 1$ and the largest integer in I_n is $n(n+1)$, it follows that each integer k with $1 \le k \le n(n+1)$ belongs to exactly one of the intervals I_m, $m = 1, 2, \ldots, n$. Hence,

$$S(n(n+1) = \sum_{m=1}^{n} \sum_{k\in I_m} \frac{1}{\langle\sqrt{k}\rangle} = \sum_{m=1}^{n} (2m)\frac{1}{m} = 2n,$$

which proves $(*)$.

Problem 4. Let S be a set of prime numbers which contains the number 2003 and has the property that for any distinct elements $q_1, q_2, \ldots, q_n$ of S, any prime factor of $q_1 q_2 \cdots q_n - 1$ belongs to S. Show that S consists of the entire set of prime numbers.

Solution. We first show that the set S must be infinite. Suppose S were finite, say $S = \{q_1, \ldots, q_n\}$, and let $P = q_1 \ldots q_n$ be the product over the primes in S. Since, by assumption, $2003 \in S$, we have $P \ge 2003$, so $P - 1$ is greater than 1 and therefore divisible by some prime q. (Without such an assumption, we could simply take $S = \{2\}$.) By the given property, q must be an element of S. However, it cannot be among the primes q_i since each of these primes divides P and therefore does not divide $P - 1$. Thus we have reached a contradiction, so S must be infinite.

Let now p be an arbitrary prime number. We need to show that p belongs to S. To this end it suffices to show that there exist primes $q_1, \ldots, q_n \in S$ such that $(*)$ $q_1 q_2 \ldots q_n - 1 \equiv 0 \bmod p$. Since S is infinite, by the pigeonhole principle, there exists a congruence class modulo p that contains infinitely many primes from S. Choose $p-1$ such primes, say $q_1, q_2, \ldots, q_{p-1}$. Then $q_1 \equiv q_2 \equiv \cdots \equiv q_{p-1} \pmod p$, and hence $q_1 \cdots q_{p-1} \equiv q_1^{p-1} \bmod p$. If $q_1 = p$, then

$p \in S$ and we are done. If $q_1 \neq p$, then, since q_1 is prime, q_1 is not congruent to 0 modulo p. Hence, by Fermat's Little Theorem, $q_1^{p-1} \equiv 1 \bmod p$, and so $(*)$ holds for the numbers $q_1, \ldots, q_{p-1}$, and the proof is complete.

Problem 5. Let $a_n = \lfloor (1+\sqrt{2})^n \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Prove that a_n is odd if n is even, and even if n is odd.

Solution. Let $\alpha = 1+\sqrt{2}$, $\beta = 1-\sqrt{2}$, and set $a_n = \alpha^n + \beta^n$. Expanding $(1 \pm \sqrt{2})^n$ by the binomial theorem, we get

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \left(\sqrt{2}^k + (-\sqrt{2})^k \right) = \sum_{k=0}^{n} \binom{n}{k} b_k,$$

say. Now,

$$b_k = \sqrt{2}^k (1 + (-1)^k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2 \cdot 2^{k/2} & \text{if } k \text{ is even,} \end{cases}$$

so in either case b_k is an even integer. Hence a_n is also an even integer. Moreover, since $\beta = 1 - \sqrt{2} = -0.41\ldots$, β^n has absolute value < 1 for all n, and is negative if n is odd, and positive if n is even. Therefore

$$\lfloor \alpha^n \rfloor = \lfloor a_n - \beta^n \rfloor = \begin{cases} a_n & \text{if } n \text{ is odd,} \\ a_n - 1 & \text{if } n \text{ is even,} \end{cases}$$

and since a_n is even for all n, it follows that $\lfloor \alpha^n \rfloor$ is even when n is odd, and odd when n is even.