UIUC Mock Putnam Exam 4/2002
Solutions

Problem 1. Without numerical calculations, determine which of the two numbers $2.01^{1.99}$ and $1.99^{2.01}$ is larger.

Solution. The first of the two numbers is the larger one. More generally, we show that $(*)$ $a^b < b^a$ whenever a and b are positive real numbers less than $e = 2.718\ldots$, with $a < b$. By taking logarithms, $(*)$ is seen to be equivalent to $b \ln a < a \ln b$, which in turn is equivalent to $(**)$ $(\ln a)/a < (\ln b)/b$. To prove $(**)$ for $0 < a < b < e$, it suffices to show that the function $f(x) = (\ln x)/x$ is increasing when $x < e$. But this is clear since $f'(x) = (1 - \ln x)x^{-2} > 0$ for $x < e$.

Problem 2. Let $f(x) = x^3 e^{x^2}(1 - x^2)^{-2}$. Find $f^{(2002)}(0)$. (Here, $f^{(n)}$ denotes the nth derivative of f.)

Solution. The answer is $f^{(2002)}(0) = 0$. A brute force approach, using the product rule, obviously won't work here. The trick is to exploit the connection between derivatives of a function and the coefficients of its Taylor series: If $f(x)$ has Taylor series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $(*)$ $a_n = f^{(n)}(0)/n!$. Now, the Taylor series of $f(x)$ is the product of the Taylor series of the three functions x^3, e^{-x^2}, and $(1 - x^2)^{-2}$. The Taylor series for the latter two functions, (namely, $\sum_{n=0}^{\infty} x^{2n}/n!$ and $\sum_{n=0}^{\infty} \binom{-2}{n} x^{2n}$) involve only *even* powers of x. Hence the product of these two series also involves only even powers of x, and multiplying with x^3 we get a power series involving only *odd* powers of x. Since this series is the Taylor series of $f(x)$, it follows that all even-indexed Taylor coefficients, and thus by $(*)$ also all values $f^{(n)}(0)$ for even n, are zero.

Problem 3. Find a formula for

$$(1^2 + 1)1! + (2^2 + 1)2! + \cdots + (n^2 + 1)n!$$

Solution. Let $S_n = \sum_{k=1}^{n} (k^2 + 1)k!$ denote the given sum. We show that $S_n = n(n + 1)!$. Let $T_n = \sum_{k=1}^{n} k!$. Using the identity $k^2 + 1 = (k + 1)(k + 2) - 3(k + 1) + 2$, we can express the given sum as a "telescoping" combination of the sums T_n:

$$\begin{aligned} S_n &= T_{n+2} - T_2 - 3(T_{n+1} - T_1) + 2T_n \\ &= T_n + (n+1)! + (n+2)! - 1 - 2 - 3(T_n + (n+1)! - 1) + 2T_n \\ &= (n+2)! - 2(n+1)! = n(n+1)! \end{aligned}$$

Problem 4. Show that $\int_0^1 x^{-x} = \sum_{n=1}^{\infty} n^{-n}$.

Solution. The trick is to write $x^x = e^{-x\ln x}$, expand the exponential into a Taylor series, and interchange the order of integration and summation:

$$\int_0^1 x^x dx = \int_0^1 e^{-x\ln x} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n (x\ln x)^n}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} I_n,$$

where $I_n = \int_0^1 x^n (\ln x)^n dx$. (The interchanging of summation and integration here is justified since the exponent, $x\ln x$, is bounded on the interval $(0,1)$ [an easy calculus exercise shows that its maximum, in absolute value, on that interval is e], and the series is therefore "uniformly convergent".) It remains to calculate the integrals I_n. A routine, though somewhat tedious, exercise in second semester calculus (involving repeated integration by parts and induction) shows that, more generally, $I_{n,m} = \int_0^1 x^n (\ln x)^m dx = (-1)^m m!(n+1)^{-m+1}$ for any nonnegative integers m and n. Thus, $I_n = (-1)^n n!(n+1)^{n+1}$. The desired formula follows upon substituting this value into the expression above.

Problem 5. Evaluate the integral

$$\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}.$$

Solution. Standard integration techniques won't get you anywhere here. However, we can take advantage of certain symmetry properties of the integrand. Let I denote the given integral. Writing $\tan x = \sin x / \cos x$ and clearing denominators gives

$$I = \int_0^{\pi/2} \frac{(\cos x)^{\sqrt{2}}}{(\cos x)^{\sqrt{2}} + (\sin x)^{\sqrt{2}}} dx.$$

Now, make the change of variables $x = \pi/2 - y$. Since $\cos(\pi/2 - y) = \sin y$ and $\sin(\pi/2 - y) = \cos y$, the integral becomes

$$I = \int_0^{\pi/2} \frac{(\sin y)^{\sqrt{2}}}{(\sin y)^{\sqrt{2}} + (\cos y)^{\sqrt{2}}} dy.$$

Adding the two formulas for I, we get $2I = \int_0^1 1 dx = \pi/2$, so $I = \pi/4$.