

UIUC Mock Putnam Exam 1/2003

Advanced Version

Problem 1. Evaluate the integral

$$I_n = \int_0^\pi \left(\frac{\sin(nx)}{\sin x} \right)^2 dx$$

for all positive integral values of n .

Problem 2. [UIUC Undergrad Math Contest '99] Define a sequence $\{x_n\}$ by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2}^{x_n}$ for $n \geq 1$. Prove that the sequence $\{x_n\}$ converges and find its limit.

Problem 3. [Putnam 1986, A2] Determine the rightmost digit (in decimal) of $\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor$. (Here $[x]$ denotes the greatest integer $\leq x$.)

Problem 4. [Putnam 1991, A2] Let A and B be different $n \times n$ matrices. If $A^3 = B^3$ and $A^2B = B^2A$, can the matrix $A^2 + B^2$ be invertible?