

UIUC Mock Putnam Exam 1/2003
Advanced Version
Solutions

Problem 1. Evaluate the integral

$$I_n = \int_0^\pi \left(\frac{\sin(nx)}{\sin x} \right)^2 dx$$

for all positive integral values of n .

Solution. We show that $I_n = n\pi$. Clearly, $I_1 = \pi$, so it suffices to show that, for $n \geq 2$, $I_n - I_{n-1} = \pi$. From the identity

$$\sin(\alpha + \beta) \sin(\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta$$

(which can be derived from the identities for $\sin(\alpha \pm \beta)$) we have $\sin^2 nx - \sin^2(n-1)x = \sin(2n-1)x \sin x$. Hence

$$I_n - I_{n-1} = \int_0^\pi \frac{\sin^2 nx - \sin^2(n-1)x}{\sin^2 x} dx = \int_0^\pi \frac{\sin(2n-1)x}{\sin x} dx = J_{2n-1},$$

say, and it suffices to show that for odd values of m , $J_m = \pi$. Using the identity

$$\sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2} = \frac{1}{2} (\sin \alpha - \sin \beta)$$

with $\alpha = (m-2)x$ and $\beta = mx$, we see that, for $m \geq 2$,

$$J_m - J_{m-2} = \int_0^\pi = 2 \int_0^\pi \frac{\sin(mx) - \sin(m-2)x}{\sin x} dx = \int_0^\pi \cos(m-1)x dx = 0.$$

Hence $J_{2n-1} = J_{2n-3} = \dots = J_1 = 2 \int_0^\pi dx = \pi$.

Problem 2. [UIUC Undergrad Math Contest '99] Define a sequence $\{x_n\}$ by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2}^{x_n}$ for $n \geq 1$. Prove that the sequence $\{x_n\}$ converges and find its limit.

Solution. Since $x_1 = \sqrt{2} < 2$ and if $x_n < 2$ then $x_{n+1} = \sqrt{2}^{x_n} < \sqrt{2}^2 = 2$, it follows by induction that (1) $x_n < 2$ for all n . Thus, the sequence $\{x_n\}$ is bounded from above. Next let $f(x) = \sqrt{2}^x - x$. Then $f'(x) = \sqrt{2}^x \log \sqrt{2} - 1 < 2 \log \sqrt{2} - 1 < 0$ for $x < 2$, so $f(x)$ is decreasing for $x < 2$, and since $f(2) = 0$, this implies $f(x) > 0$, or equivalently $\sqrt{2}^x > x$, for $x < 2$. In view of (1), it follows that $x_{n+1} = \sqrt{2}^{x_n} > x_n$ for all n . Hence the sequence $\{x_n\}$ is monotone increasing and bounded from above and therefore must be convergent. Let L denote the limit of this sequence. By (1) we have (2) $L = \lim_{n \rightarrow \infty} x_n \leq 2$, and letting $n \rightarrow \infty$ on both sides of the recurrence $x_{n+1} = \sqrt{2}^{x_n}$, we obtain $L = \sqrt{2}^L$ or (3) $f(L) = 0$. Since $f(2) = 0$, $L = 2$ is a solution to (3). Moreover, $L = 2$ is the only solution satisfying (2), since $f(x)$ is decreasing for $x < 2$. Hence the limit of the sequence $\{x_n\}$ is 2.

Problem 3. [Putnam 1986, A2] Determine the rightmost digit (in decimal) of $\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor$. (Here $[x]$ denotes the greatest integer $\leq x$.)

Solution. Let x denote the number in brackets. Expanding $(1+3 \cdot 10^{-100})^{-1}$ into a geometric series, we obtain

$$x = \sum_{n=0}^{\infty} (-1)^n 3^n 10^{19,900-100n}.$$

In the last sum, all terms with $n < 199$ are all divisible by 10 and the term $n = 199$ equals $(-3)^{199}$. Also, since the series is alternating with decreasing terms, the sum of the terms with $n \geq 200$ is positive and bounded from above by the first of these terms, i.e., $3^{200}10^{-100}$, which is less than 1. Thus, the last digit of $[x]$ is equal to the last digit of $N = (-3)^{199}$ or, equivalently, the residue of N modulo 10. Since $(-3)^4 \equiv 1$ modulo 10, we have $(-3)^{199} \equiv (-3)^3 \equiv 3$ modulo 10, so the rightmost digit of $[x]$ is 3.

Problem 4. [Putnam 1991, A2] Let A and B be different $n \times n$ matrices. If $A^3 = B^3$ and $A^2B = B^2A$, can the matrix $A^2 + B^2$ be invertible?

Solution. The answer is no. To prove that $A^2 + B^2$ is **not** invertible, it suffices to find a non-zero matrix C such that $(A^2 + B^2)C$ is the zero matrix. (If $A^2 + B^2$ had an inverse D , then multiplying the equation $(A^2 + B^2)C = 0$ from the left by D would give $C = 0$, a contradiction.) We show that $C = A - B$ (which is non-zero, since by assumption A and B are **different** matrices) has this property:

$$(A^2 + B^2)(A - B) = A^2A + B^2A - A^2B - B^2B = A^3 - B^3 + B^2A - A^2B = 0.$$