

UIUC Mock Putnam Exam 1/2003  
Advanced Version  
Solutions

**Problem 1.** Evaluate the integral

$$I_n = \int_0^\pi \left( \frac{\sin(nx)}{\sin x} \right)^2 dx$$

for all positive integral values of  $n$ .

**Solution.** We show that  $I_n = n\pi$ . Clearly,  $I_1 = \pi$ , so it suffices to show that, for  $n \geq 2$ ,  $I_n - I_{n-1} = \pi$ . From the identity

$$\sin(\alpha + \beta) \sin(\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta$$

(which can be derived from the identities for  $\sin(\alpha \pm \beta)$ ) we have  $\sin^2 nx - \sin^2(n-1)x = \sin(2n-1)x \sin x$ . Hence

$$I_n - I_{n-1} = \int_0^\pi \frac{\sin^2 nx - \sin^2(n-1)x}{\sin^2 x} dx = \int_0^\pi \frac{\sin(2n-1)x}{\sin x} dx = J_{2n-1},$$

say, and it suffices to show that for odd values of  $m$ ,  $J_m = \pi$ . Using the identity

$$\sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2} = \frac{1}{2} (\sin \alpha - \sin \beta)$$

with  $\alpha = (m-2)x$  and  $\beta = mx$ , we see that, for  $m \geq 2$ ,

$$J_m - J_{m-2} = \int_0^\pi = 2 \int_0^\pi \frac{\sin(mx) - \sin(m-2)x}{\sin x} dx = \int_0^\pi \cos(m-1)x dx = 0.$$

Hence  $J_{2n-1} = J_{2n-3} = \cdots = J_1 = 2 \int_0^\pi dx = \pi$ .

**Problem 2.** [UIUC Undergrad Math Contest '99] Define a sequence  $\{x_n\}$  by  $x_1 = \sqrt{2}$  and  $x_{n+1} = \sqrt{2}^{x_n}$  for  $n \geq 1$ . Prove that the sequence  $\{x_n\}$  converges and find its limit.

**Solution.** Since  $x_1 = \sqrt{2} < 2$  and if  $x_n < 2$  then  $x_{n+1} = \sqrt{2}^{x_n} < \sqrt{2}^2 = 2$ , it follows by induction that (1)  $x_n < 2$  for all  $n$ . Thus, the sequence  $\{x_n\}$  is bounded from above. Next let  $f(x) = \sqrt{2}^x - x$ . Then  $f'(x) = \sqrt{2}^x \log \sqrt{2} - 1 < 2 \log \sqrt{2} - 1 < 0$  for  $x < 2$ , so  $f(x)$  is decreasing for  $x < 2$ , and since  $f(2) = 0$ , this implies  $f(x) > 0$ , or equivalently  $\sqrt{2}^x > x$ , for  $x < 2$ . In view of (1), it follows that  $x_{n+1} = \sqrt{2}^{x_n} > x_n$  for all  $n$ . Hence the sequence  $\{x_n\}$  is monotone increasing and bounded from above and therefore must be convergent. Let  $L$  denote the limit of this sequence. By (1) we have (2)  $L = \lim_{n \rightarrow \infty} x_n \leq 2$ , and letting  $n \rightarrow \infty$  on both sides of the recurrence  $x_{n+1} = \sqrt{2}^{x_n}$ , we obtain  $L = \sqrt{2}^L$  or (3)  $f(L) = 0$ . Since  $f(2) = 0$ ,  $L = 2$  is a solution to (3). Moreover,  $L = 2$  is the only solution satisfying (2), since  $f(x)$  is decreasing for  $x < 2$ . Hence the limit of the sequence  $\{x_n\}$  is 2.

**Problem 3.** [Putnam 1986, A2] Determine the rightmost digit (in decimal) of  $\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor$ . (Here  $[x]$  denotes the greatest integer  $\leq x$ .)

**Solution.** Let  $x$  denote the number in brackets. Expanding  $(1+3 \cdot 10^{-100})^{-1}$  into a geometric series, we obtain

$$x = \sum_{n=0}^{\infty} (-1)^n 3^n 10^{19,900-100n}.$$

In the last sum, all terms with  $n < 199$  are all divisible by 10 and the term  $n = 199$  equals  $(-3)^{199}$ . Also, since the series is alternating with decreasing terms, the sum of the terms with  $n \geq 200$  is positive and bounded from above by the first of these terms, i.e.,  $3^{200}10^{-100}$ , which is less than 1. Thus, the last digit of  $[x]$  is equal to the last digit of  $N = (-3)^{199}$  or, equivalently, the residue of  $N$  modulo 10. Since  $(-3)^4 \equiv 1$  modulo 10, we have  $(-3)^{199} \equiv (-3)^3 \equiv 3$  modulo 10, so the rightmost digit of  $[x]$  is 3.

**Problem 4.** [Putnam 1991, A2] Let  $A$  and  $B$  be different  $n \times n$  matrices. If  $A^3 = B^3$  and  $A^2B = B^2A$ , can the matrix  $A^2 + B^2$  be invertible?

**Solution.** The answer is no. To prove that  $A^2 + B^2$  is **not** invertible, it suffices to find a non-zero matrix  $C$  such that  $(A^2 + B^2)C$  is the zero matrix. (If  $A^2 + B^2$  had an inverse  $D$ , then multiplying the equation  $(A^2 + B^2)C = 0$  from the left by  $D$  would give  $C = 0$ , a contradiction.) We show that  $C = A - B$  (which is non-zero, since by assumption  $A$  and  $B$  are **different** matrices) has this property:

$$(A^2 + B^2)(A - B) = A^2A + B^2A - A^2B - B^2B = A^3 - B^3 + B^2A - A^2B = 0.$$