UIUC Mock Putnam Exam 2/2003
Advanced Version
Solutions

Problem 1. Evaluate the integral

$$I = \int_0^\pi \ln(\sin x)dx.$$

Solution. Setting $x = 2y$, and using the double angle formula, the integrand can be written as

$$\ln(\sin x) = \ln(2 \sin y \cos y) = \ln 2 + \ln(\sin y) + \ln(\cos y).$$

Hence

$$I = \pi \ln 2 + 2 \int_0^{\pi/2} \ln(\sin y)dy + 2 \int_0^{\pi/2} \ln(\cos y)dy = \pi \ln 2 + 2I_1 + 2I_2,$$

say. Changing variables $y = \pi/2 - z$ and using the identity $\cos(\pi/2 - z) = \sin z$, we see that the second integral on the right, I_2, is equal to the first, and so $I = \pi \ln 2 + 4I_1$. Moreover, since $\sin(\pi - y) = \sin y$, we have

$$I_1 = \frac{1}{2} \int_0^{\pi/2} (\ln(\sin y) + \ln(\sin(\pi - y)))dy = \frac{1}{2} I.$$

Thus, $I = \pi \ln 2 + 4(1/2)I$, and so $I = -\pi \ln 2$.

Problem 2. Let a, b, c denote distinct integers. Does there exist a polynomial with integral coefficients that cyclically permutes these values in the sense that $P(a) = b$, $P(b) = c$ and $P(c) = a$? Explain!

Solution. The answer is no. For suppose that $P(x)$ is a polynomial with the given property. Then a is a root of $P(x) - b$, and so by the factor theorem $P(x) - b = (x - a)Q(x)$, where $Q(x)$ is a polynomial with integer coefficients.

In particular, the number $(c-b)/(b-a) = (P(b)-b)/(b-a) = Q(b)$ must be an integer. Similarly, each of the numbers $(a-c)/(c-b)$ and $(b-a)/(a-c)$ must also be an integer. Multiplying these three numbers we get

$$\frac{c-b}{b-a} \cdot \frac{a-c}{c-b} \cdot \frac{b-a}{a-c} = 1,$$

which is only possible if each of the numbers is 1 or -1, i.e., if $|c-b| = |a-b| = |a-c|$. But the latter is clearly impossible for distinct integers a, b, c.

Problem 3. [Putnam 1966, A3] Let $0 < x_0 < 1$, and $x_{n+1} = x_n(1-x_n)$ for $n \geq 0$. Prove that the limit $\lim_{n\to\infty} nx_n$ exists and is equal to 1.

Solution. Note first that the given relation implies, by an easy induction argument, $0 < x_n < 1$ for all n. Next, let $y_n = 1/x_n$. Then y_n satisfies the recurrence relation

$$y_{n+1} = \frac{1}{x_n(1-x_n)} = \frac{1}{x_n} + \frac{1}{1-x_n} = y_n + \frac{1}{1-x_n}$$

for all n, together with the initial condition $y_0 = 1/x_0 > 1$. Since $0 < x_n < 1$ for all n, this implies $y_{n+1} > y_n + 1$ for all n, and hence, by induction, $y_{n+1} > y_0 + (n+1) > n+2$. Thus, $x_n < 1/(n+1)$ for all n, and inserting this bound into the above identity for y_{n+1} gives

$$y_{n+1} < y_n + \left(1 - \frac{1}{n+1}\right)^{-1} = y_n + 1 + \frac{1}{n}$$

for $n \geq 1$. Hence, again by induction,

$$y_{n+1} < y_1 + n + \sum_{k=1}^{n} \frac{1}{k}.$$

The latter sum is at most $1 + \ln n$ (as can be seen by comparing this sum to the integral $\int_1^n (1/x)dx = \ln x$). Thus, we have

$$n + 2 < y_{n+1} < n + \ln n + 1 + y_1,$$

Dividing by $n+1$ and letting $n \to \infty$, we obtain $\lim_{n\to\infty} y_{n+1}/(n+1) = 1$. Since $nx_n = n/y_n$, it follows that $\lim_{n\to\infty} nx_n = 1$.

Problem 4. Evaluate $\sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k$.

Solution. We use generating functions. Writing $\binom{n}{k}^2 = \binom{n}{k}\binom{n}{n-k}$, we see that the given sum is the coefficient of x^n in the polynomial

$$\left(\sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k\right) \left(\sum_{m=0}^{n} \binom{n}{m} x^m\right),$$

which by the binomial theorem can be written as

$$(1 + (-x))^n (1 + x)^n = (1 - x^2)^n.$$

Expanding $(1 - x^2)^n$ by the binomial theorem, we see that the power x^n does not appear if n is odd, and if n is even, it appears with coefficient $\binom{n}{n/2}(-1)^{n/2}$. Hence the given binomial sum is equal to $\binom{n}{n/2}(-1)^{n/2}$ if n is even, and zero otherwise.