UIUC Mock Putnam Exam 3/2003
Advanced Version

Problem 1. A *binary partition* of an integer n is a partition of n into parts of the form 2^i, $i = 0, 1, \ldots$ (with repetition allowed). For example, $n = 5$ has the following binary partitions: $5 = 4 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1$. Let $b(n)$ denote the number of binary partitions of n; the first few values of $b(n)$ are $b(1) = 1$, $b(2) = 2$, $b(3) = 2$, $b(4) = 4$, $b(5) = 4$. Show that $b(n)$ is even for all $n \geq 2$.

Problem 2. Let $P(x)$ be a polynomial of degree n satisfying $P(k) = 1/k$ for $k = 1, 2, \ldots, n + 1$. What is $P(n + 2)$?

Problem 3. [1995 UIUC Undergrad Math Contest] Let $f(n)$ be a nonnegative function, defined on the set of nonnegative integers and satisfying $f(n + m) \leq f(n) + f(m)$ for all n, m. Show that $\lim_{n\to\infty} f(n)/n$ exists.

Problem 4. [1995 UIUC Undergrad Math Contest] Let c be a positive constant, let $0 < x_1 < x_0 < 1$, and for $n \geq 1$ let $x_{n+1} = cx_n x_{n-1}$. Prove that there exists a positive real number α such that the limit $L = \lim_{n\to\infty} x_{n+1}/x_n^\alpha$ exists and $0 < L < \infty$.