UIUC Mock Putnam Exam 3/2003
Advanced Version
Solutions

Problem 1. A *binary partition* of an integer n is a partition of n into parts of the form 2^i, $i = 0, 1, \dots$ (with repetition allowed). For example, $n = 5$ has the following binary partitions: $5 = 4 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1$. Let $b(n)$ denote the number of binary partitions of n; the first few values of $b(n)$ are $b(1) = 1$, $b(2) = 2$, $b(3) = 2$, $b(4) = 4$, $b(5) = 4$. Show that $b(n)$ is even for all $n \geq 2$.

Solution. We first establish a recurrence for $b(n)$. Let $b_k(n)$ denote the number of binary partitions with exactly k 1's. Then $b(n) = \sum_{k=0}^{n} b_k(n)$. Furthermore, $b_n(n) = 1$, and if $k < n$, the number of binary partitions of n with exactly k 1's is equal to the number of binary partitions of $n - k$ containing no 1's, i.e., the number of partitions of $n - k$ into parts of the form $2, 2^2, \dots$. But the latter number is only nonzero if $n - k$ is even, and dividing through by 2 we see that in this case it is equal to the number of unrestricted binary partitions of $(n - k)/2$. Thus, for all $k = 0, \dots, n - 1$ we have $b_k(n) = b((n - k)/2)$ where $b(x)$ is to be interpreted as 0 when x is not a positive integer. It follows that for $n \geq 4$

$$\begin{aligned} b(n) &= \sum_{k=0}^{n-1} b_k(n) + b_n(n) = \sum_{k=0}^{n-1} b((n-k)/2) + 1 \\ &= \sum_{m=1}^{[n/2]} b(m) + 1 = 2 + \sum_{m=2}^{[n/2]} b(m). \end{aligned}$$

This shows that if the terms $b(m)$, $m = 2, 3, \dots, [n/2]$, are all even, then so is $b(n)$. Since $b(2) = b(3) = 2$, induction gives that $b(n)$ is even for all $n \geq 2$.

Problem 2. Let $P(x)$ be a polynomial of degree n satisfying $P(k) = 1/k$ for $k = 1, 2, \dots, n + 1$. What is $P(n + 2)$?

Solution. Let $Q(x) = xP(x) - 1$. Then $Q(x)$ is a polynomial of degree $n+1$ with zeros at $k = 1, 2, \dots, n+1$. By the factor theorem, $Q(x)$ must of the form $(*)$ $Q(x) = c\prod_{k=1}^{n+1}(x-k)$, where c is a constant (depending on n). Also, $Q(0) = 0 \cdot P(0) - 1 = -1$, and setting $x = 0$ in $(*)$ gives $-1 = c\prod_{k=1}^{n+1}(-k) = c(-1)^{n+1}(n+1)!$, so $c = (-1)^n/(n+1)!$. Hence $Q(n+2) = c(n+1)! = (-1)^n$, so $P(n+2) = (1+Q(n+2))/(n+2) = (1+(-1)^n)/(n+2)$, i.e., $P(n+2) = 0$ if n is odd, and $P(n+2) = 2/(n+2)$ if n is even.

Problem 3. [1995 UIUC Undergrad Math Contest] Let $f(n)$ be a nonnegative function, defined on the set of nonnegative integers and satisfying $f(n+m) \le f(n) + f(m)$ for all n, m. Show that $\lim_{n\to\infty} f(n)/n$ exists.

Solution. First, taking $n = m = 0$ in the given inequality, we obtain $f(0) \le 2f(0)$, which forces $f(0) = 0$. Next, let m be a positive integer. From the given inequality we obtain $f(km) = f(m+(k-1)m) \le f(m)+f((k-1)m)$ for all positive integers k. By induction, it follows that

$$f(km) \le kf(m) \quad \text{for all } k \qquad (*)$$

Now, set $g(n) = f(n)/n$ for $n \ge 1$. Then $(*)$ implies $g(n) \le g(m)$ for all positive integers n that are multiples of m. On the other hand, if $n = km+r$ with $0 < r < m$, then, using the given inequality we have

$$g(n) = \frac{f(n)}{n} \le \frac{f(km) + f(r)}{n} = \frac{kmg(km) + rg(r)}{n} < g(m) + \frac{C_m}{n},$$

where $C_m = \max_{r=1}^{m-1} rg(r)$ is a constant depending only on m. It follows that

$$\limsup_{n\to\infty} g(n) \le g(m).$$

Since m was arbitray, and the left-hand side does not involve m, we have

$$\limsup_{n\to\infty} g(n) \le \inf_m g(m) \le \liminf_{m\to\infty} g(m).$$

But this implies that the limsup and liminf of $g(n)$ are equal, i.e., that $\lim_{n\to\infty} g(n)$ exists, which is what we had to prove.

Problem 4. [1995 UIUC Undergrad Math Contest] Let c be a positive constant, let $0 < x_1 < x_0 < 1$, and for $n \ge 1$ let $x_{n+1} = cx_nx_{n-1}$. Prove that there exists a positive real number α such that the limit $L = \lim_{n\to\infty} x_{n+1}/x_n^\alpha$ exists and $0 < L < \infty$.

Solution. We transform the sequence into one that satisfies the Fibonnacci recursion. To this end, let first $y_n = cx_n$. Then y_n satisfies the recursion

$y_{n+1} = y_n y_{n-1}$ for $n \geq 1$, along with the initial condition $0 < y_1 < y_0 < 1$. Setting $a_n = -\ln y_n$, we then have $a_{n+1} = a_n + a_{n-1}$ for $n \geq 1$ and $a_1 > a_0 > 0$. The latter recurrence is the same as that satisfied by the Fibonnacci numbers. Its characteristic equation, $x^2 - x - 1 = 0$, has roots at $x = \phi$ and $x = \overline{\phi}$, where $\phi = (1 + \sqrt{5})/2$ and $\overline{\phi} = (1 - \sqrt{5})/2$. By the general theory of second order linear recurrences, any solution to this recurrence is of the form $a_n = c_1\phi^n + c_2\overline{\phi}^n$ with coefficients c_1 and c_2. Since $|\overline{\phi}| < 1$, it follows that, as $n \to \infty$, $a_n - c_1\phi^n \to 0$ and therefore $a_{n+1} - \phi a_n \to 0$. Hence $y_{n+1}/y_n^\phi \to 1$, and substituting $y_n = cx_n$, we get $x_{n+1}/x_n^\phi \to c^{\phi-1}$ as $n \to \infty$. Thus, the limit L sought in the problem exists and is equal to $c^{\phi-1}$.